

# EXTENSION OF SYMMETRIES ON EINSTEIN MANIFOLDS WITH BOUNDARY

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**ABSTRACT.** We investigate the validity of the isometry extension property for (Riemannian) Einstein metrics on compact manifolds  $M$  with boundary  $\partial M$ . Given a metric  $\gamma$  on  $\partial M$ , this is the issue of whether any Killing field  $X$  of  $(\partial M, \gamma)$  extends to a Killing field of any Einstein metric  $(M, g)$  bounding  $(\partial M, \gamma)$ . Under a mild condition on the fundamental group, this is proved to be the case at least when  $X$  preserves the mean curvature of  $\partial M$  in  $(M, g)$ .

## 1. INTRODUCTION.

Let  $M^{n+1}$  be a compact  $(n+1)$ -dimensional manifold-with-boundary, and suppose  $g$  is a (Riemannian) Einstein metric on  $M$ , so that

$$(1.1) \quad Ric_g = \lambda g,$$

for some constant  $\lambda \in \mathbb{R}$ . The metric  $g$  induces a Riemannian boundary metric  $\gamma$  on  $\partial M$ . In this paper we consider the issue of whether isometries of the boundary structure  $(\partial M, \gamma)$  necessarily extend to isometries of any filling Einstein manifold  $(M, g)$ .

In general, without any assumptions, this isometry extension property will not hold. It is false for instance if  $\partial M$  is not connected. For example, let  $M = S^3 \setminus (B_1 \cup B_2)$ , where  $B_i$  are a pair of disjoint round 3-balls in  $S^3$  endowed with a round metric; then a generic pair of Killing fields  $X_i$  on  $S_i^2 = \partial B_i$  does not extend to a Killing field on  $M$ . Also, setting  $M = T^3 \setminus B$  where  $B$  is a round 3-ball in a flat 3-torus  $T^3$ , one sees again that Killing fields on  $\partial M$  do not extend to Killing fields on  $T^3$ . This is due to the fact that  $\pi_1(\partial M)$  does not surject onto  $\pi_1(M)$ . Both situations above can be remedied by making the topological assumption

$$(1.2) \quad \pi_1(M, \partial M) = 0,$$

so we will usually assume (1.2).

However, this condition is still not sufficient. Consider for example the flat product metric on  $S^1 \times \mathbb{R}^2$ . Let  $\sigma$  be any simple closed curve in  $\mathbb{R}^2$  and let  $T_\sigma = S^1 \times \sigma \subset S^1 \times \mathbb{R}^2$ . Then  $T_\sigma$  bounds a compact domain  $M \subset S^1 \times \mathbb{R}^2$ , diffeomorphic to a solid torus. Any such  $T_\sigma$  is flat with respect to the induced metric, and so has a pair of orthogonal Killing fields. One of these, that tangent to the  $S^1$  factor, clearly extends to a Killing field of  $M$  (in fact  $S^1 \times \mathbb{R}^2$ ). However, whenever  $\sigma$  is not a round circle in  $\mathbb{R}^2$  (so that  $\sigma$  has non-constant geodesic curvature) the orthogonal Killing field on  $(T_\sigma, \gamma)$  tangent to  $\sigma$  does not extend as a Killing field to  $M$ .

Very similar examples are easily constructed via the Hopf fibration in the sphere  $S^3$ , with  $M$  again a solid torus in  $S^3$ , as first pointed out to the author by H. Rosenberg [15], cf. [7], [10], [14] and references therein for detailed discussion. Similar examples, even with convex boundary, also occur in hyperbolic space-forms, cf. Remark 3.6 below, and in higher dimensions by taking products.

The main result of this paper characterizes one situation where the isometry extension property does hold. Let  $H$  denote the mean curvature of  $\partial M$  in  $(M, g)$ .

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**Theorem 1.1.** *Let  $g$  be a  $C^{m,\alpha}$  Einstein metric on  $M$ ,  $m \geq 5$ , with induced boundary metric  $\gamma$  on  $\partial M$ , and suppose (1.2) holds. Then any Killing field  $X$  on  $(\partial M, \gamma)$  for which  $X(H) = 0$ , extends uniquely to a Killing field on  $(M, g)$ .*

It follows for instance that for  $H = \text{const}$ , the identity component  $Isom_0(\partial M, \gamma)$  of the isometry group of  $(\partial M, \gamma)$  embeds in the isometry group of any Einstein filling metric  $(M, g)$ :

$$Isom_0(\partial M, \gamma) \hookrightarrow Isom_0(M, g),$$

or equivalently, such isometries of the boundary extend to isometries of any Einstein filling metric. A simple consequence of Theorem 1.1 is for example the following rigidity result.

**Corollary 1.2.** *Let  $g$  be a  $C^{5,\alpha}$  Einstein metric on  $M^{n+1}$  which induces the round metric  $\gamma_{+1}$  on the boundary  $\partial M = S^n$ ,  $n \geq 2$ . If  $\pi_1(M) = 0$  and  $H = \text{const}$ , then  $(M, g)$  is isometric to a standard round ball in a simply connected space form.*

There are natural analogs of these results valid for exterior domains. Thus, let  $M^{n+1}$  be an open or non-compact manifold with compact “inner” boundary and with a finite number of non-compact ends. Metrically, consider complete metrics  $g$  on  $M$  which are asymptotically (locally) flat on each end. In this context, Theorem 1.1 also holds for Einstein metrics, cf. Proposition 5.3. A similar result also holds for complete, asymptotically hyperbolic Einstein metrics, with boundary at infinity, without any assumption on the mean curvature, cf. Theorem 5.4.

We point out that Theorem 1.1 (and Corollary 1.2) remain valid without the hypothesis (1.2) provided  $(M, g)$  is embedded as a domain in a complete, simply connected Einstein manifold  $(\hat{M}, \hat{g})$ . It should also be noted that the isometry extension property is false for isometries not contained in  $Isom_0(\partial M, \gamma)$ . As a simple example, consider a flat metric on a solid torus  $M = D^2 \times S^1$  of the form

$$g_0 = dr^2 + r^2 d\theta_1^2 + d\theta_2^2,$$

for  $r \in [0, 1]$ . Then interchanging the two circles parametrized by  $\theta_1$  and  $\theta_2$  is an isometry of the boundary, which does not extend to an isometry of the solid torus. Of course  $\partial M$  is both convex and has constant mean curvature in  $(M, g_0)$ .

The proofs of the results above follow from a study of the global properties of the space of Einstein metrics  $g$  on  $M$ . As shown in [3], the moduli space  $\mathcal{E}$  of such metrics is a smooth Banach manifold, for which the (Dirichlet) map to the boundary metrics

$$(1.3) \quad \Pi_D : \mathcal{E} \rightarrow \text{Met}(\partial M), \quad \Pi_D(g) = g_{T(\partial M)},$$

is  $C^\infty$  smooth, cf. Theorem 2.1. The main results are then quite simple to prove when the metric  $(M, g)$  is non-degenerate, in the strong sense that the derivative  $D\Pi_D$  of  $\Pi_D$  at  $g$  has trivial kernel, cf. Remark 3.3. They also hold, with somewhat more involved proofs, when  $D\Pi_D$  has no cokernel, or more precisely when  $\text{Im} D\Pi_D$  is dense in  $T\text{Met}(\partial M)$ , cf. Proposition 4.4. As discussed in §3, note however that the map  $\Pi_D$  is never Fredholm, and the image of the linearization is always of infinite codimension. In both of the situations above, the results hold without any condition on the mean curvature, i.e. without assuming  $X(H) = 0$ .

In general, the strategy is to prove the implication

$$(1.4) \quad X(H) = \mathcal{L}_X H = 0 \Rightarrow \mathcal{L}_X A = 0,$$

where  $A$  is the 2<sup>nd</sup> fundamental form of  $\partial M$  in  $M$  and  $\mathcal{L}_X$  is the Lie derivative with respect to  $X$ . Given this, Theorem 1.1 then follows from a unique continuation theorem for Einstein metrics proved in [2]. A key point is to relate (1.4) with the linearization of the divergence constraint for the Einstein equations at  $\partial M$ , which reads:

$$(1.5) \quad \delta'_h(A - H\gamma) + \delta(A - H\gamma)'_h = -(\text{Ric}(N, \cdot))'_h \quad \text{at } \partial M,$$

where  $N$  is the unit normal and  $\delta$  is the divergence operator. We show in §4 (cf. Lemma 4.2) that (1.4) holds provided the linearized divergence constraint for the Einstein equations is “surjective” at  $\partial M$ . This means that any symmetric form  $h^T$  on  $\partial M$  has an extension to a neighborhood of  $\partial M$  in  $M$  such that the derivative

$$(1.6) \quad (Ric'_h)(N, \cdot) = 0 \quad \text{at } \partial M,$$

This is of course closely related to the surjectivity of  $D\Pi_D$ . Now while (1.6) does not hold in general (i.e. for all  $h^T$  on  $\partial M$ ) we prove that any  $h$  as above always has an extension such that

$$(1.7) \quad \int_{\partial M} Ric'_h(N, \cdot) dV_\gamma = 0,$$

and this suffices to establish (1.4). The proof of (1.7) requires a careful study of the linearized Einstein operator, and related operators, with certain self-adjoint, elliptic boundary conditions distinct (of course) from Dirichlet boundary data; see in particular the operators and boundary conditions in (5.1) and (5.16).

A brief survey of the contents of the paper is as follows. In §2, we introduce the basic setting and structural results on the space of Einstein metrics, needed for the work to follow. Section 3 studies elliptic boundary value problems for the Einstein equations and the lack of the Fredholm property for the boundary map  $\Pi_D$  in (1.3). Section 4 relates the isometry extension property with the linearized divergence constraint equations induced by the Einstein equations on  $\partial M$ . In §5, we prove Theorem 1.1 and Corollary 1.2, and the further related results mentioned above.

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## 2. THE SPACE OF EINSTEIN METRICS

As above, let  $M$  denote a connected, compact, oriented  $(n+1)$ -dimensional manifold with compact, non-empty boundary  $\partial M$ . Consider the Banach space

$$(2.1) \quad Met(M) = Met^{m,\alpha}(M)$$

of Riemannian metrics on  $M$  which are  $C^{m,\alpha}$  smooth up to  $\partial M$ . Here  $m$  is any fixed integer with  $m \geq 2$ , including  $m = \infty$  (giving a Fréchet space) and  $\alpha \in (0, 1)$ . Let

$$(2.2) \quad \mathbb{E} = \mathbb{E}^{m,\alpha}(M) \subset Met^{m,\alpha}(M)$$

be the subset of Einstein metrics on  $M$ ,  $C^{m,\alpha}$  smooth up to  $\partial M$ , with

$$(2.3) \quad Ric_g = \lambda g,$$

for  $\lambda$  arbitrary, but fixed (so that  $\mathbb{E} = \mathbb{E}(\lambda)$ );  $Ric_g$  is the Ricci curvature of  $g$ . The smoothness index  $(m, \alpha)$  will occasionally be suppressed from the notation when its exact value is unimportant.

The space  $\mathbb{E}^{m,\alpha}(M) \subset Met^{m,\alpha}(M)$  is invariant under the action of the group  $\mathcal{D}_1 = \mathcal{D}_1^{m+1,\alpha}$  of orientation preserving  $C^{m+1,\alpha}$  diffeomorphisms of  $M$  equal to the identity on  $\partial M$ . This action is free (since any such isometry equal to the identity on  $\partial M$  is necessarily the identity) and well-known to be proper. The moduli space  $\mathcal{E} = \mathcal{E}^{m,\alpha}(M)$  of Einstein metrics on  $M$  is defined to be the quotient

$$(2.4) \quad \mathcal{E} = \mathbb{E}/\mathcal{D}_1.$$

One has a natural Dirichlet boundary map

$$(2.5) \quad \Pi_D : \mathbb{E} \rightarrow \text{Met}(\partial M); \quad \Pi_D(g) = \gamma = g|_{T(\partial M)}.$$

which clearly descends to a map

$$(2.6) \quad \Pi_D : \mathcal{E} \rightarrow \text{Met}(\partial M); \quad \Pi_D([g]) = \gamma.$$

We note the following result, proved in [3].

**Theorem 2.1.** *Suppose  $\pi_1(M, \partial M) = 0$  and  $m \geq 5$ . Then the space  $\mathcal{E}$  is a  $C^\infty$  smooth Banach manifold, (Fréchet manifold when  $m = \infty$ ), and the boundary map  $\Pi_D$  is  $C^\infty$  smooth.*

Theorem 2.1 is proved by a suitable application of the implicit function theorem. Strictly speaking, this result is not needed for the proof of the main results in the Introduction; however it places the arguments to follow in a natural context.

Consider the Einstein operator

$$(2.7) \quad \begin{aligned} E : \text{Met}(M) &\rightarrow S_2(M), \\ E(g) &= \text{Ric}_g - \lambda g, \end{aligned}$$

where  $S_2(M)$  is the space of symmetric bilinear forms on  $M$ . The linearization of  $E$  is given by

$$(2.8) \quad L_E(k) = 2 \frac{d}{dt} (\text{Ric}_{g+tk} - \lambda(g+tk))|_{t=0} = \nabla^* \nabla k - 2R(k) - 2\delta^* \beta(k);$$

here  $\delta^* X = \frac{1}{2} \mathcal{L}_X g$ ,  $\beta(k) = \delta(k) + \frac{1}{2} dtr k$  is the Bianchi operator with respect to  $g$ ,  $\nabla^* \nabla$  is the rough Laplacian ( $\nabla^* \nabla = -\nabla_{e_i} \nabla_{e_i}$ ) and  $R(h)$  is the action of the curvature tensor on symmetric bilinear forms  $k$ , cf. [5] for instance.

The tangent space  $T_g \mathbb{E}$  is given by  $\text{Ker} L_E$ . The derivative of the Dirichlet boundary map  $\Pi_D$  in (2.5) acts on forms  $k$  satisfying  $L_E(k) = 0$  and is given by

$$(2.9) \quad (D\Pi_D)_g(k) = k^T|_{\partial M},$$

where  $k^T$  is the tangential projection or restriction of  $k$  to  $T(\partial M)$ . Thus  $k^T$  is the variation of the boundary metric  $\gamma = \Pi_D(g)$ . It will also be important to consider the variation of the 2<sup>nd</sup> fundamental form  $A$  of  $\partial M$  in  $M$ . Thus, analogous to (2.6), one has a natural Neumann boundary map

$$(2.10) \quad \Pi_N : \mathcal{E} \rightarrow S^2(\partial M), \quad \Pi_N([g]) = A.$$

This is well-defined, since  $A$  is invariant under the action of  $\mathcal{D}_1$ . Note also that  $\Pi_N$  maps  $\mathcal{E}^{m,\alpha}$  to  $S_2^{m-1,\alpha}(\partial M)$ . To compute the derivative of  $\Pi_N$ , let  $g_s = g + sk$  be a variation of  $g$ . Since  $A = \frac{1}{2} \mathcal{L}_N g$ , one has  $2A'_k \equiv 2 \frac{d}{ds} A_{g_s}|_{s=0} = (\mathcal{L}_{N_s} g_s)'|_{s=0} = \mathcal{L}_N k + \mathcal{L}_{N'} g$ . A simple computation gives  $N' = -k(N)^T - \frac{1}{2} k_{00} N$ , where  $k(N)^T$  is the component of  $k(N)$  tangent to  $\partial M$  and  $k_{00} = k(N, N)$ . Thus

$$(2.11) \quad A'_k = (D\Pi_N)(k) = \frac{1}{2} (\mathcal{L}_N k + \delta^* V),$$

where  $V = 2N' = -2k(N)^T - k_{00} N$ .

The kernel of  $D\Pi_D$  in (2.5) consists of forms  $k$  satisfying  $L_E(k) = 0$  and  $k^T = 0$  on  $\partial M$ , while the kernel of  $D\Pi_N$  in (2.10) consists of such forms satisfying  $(A'_k)^T = 0$  at  $\partial M$ . Thus, if both conditions hold,

$$(2.12) \quad k^T = 0, \quad (A'_k)^T = 0 \quad \text{at } \partial M,$$

then  $(M, g)$  is both Dirichlet and Neumann degenerate, i.e. a singular point of each boundary map. We note that each of the conditions in (2.12) is gauge-invariant, i.e. invariant under the addition of terms of the form  $\delta^* Z$  with  $Z = 0$  on  $\partial M$ . Of course any form  $k$  satisfying  $k = \nabla_N k = 0$  at

$\partial M$  satisfies (2.12). Changing such  $k$  by arbitrary such gauge transformations shows that (2.12) is equivalent to the statement that  $k$  is pure gauge, to first order at  $\partial M$ , i.e.

$$(2.13) \quad k = \delta^* Z + O(t^2),$$

near  $\partial M$ , with  $Z = 0$  on  $\partial M$ , where  $t(x) = \text{dist}_g(x, \partial M)$ .

The natural or geometric Cauchy data for the Einstein equations (2.3) on  $M$  at  $\partial M$  consist of the pair  $(\gamma, A)$ . If  $k$  is an infinitesimal Einstein deformation of  $(M, g)$ , so that  $L_E(k) = 0$ , then the induced variation of the Cauchy data on  $\partial M$  is given by  $k^T$  and  $(A'_k)^T$ . It is natural to expect that an Einstein metric  $g$  is uniquely determined in a neighborhood of  $\partial M$ , up to isometry, by the Cauchy data  $(\gamma, A)$ , i.e. one should have a suitable unique continuation property for Einstein metrics. Similarly, one would expect this holds for the linearized Einstein equations. The next result, proved in [2] confirms this expectation; this result is also proved in [6], but only under the much stronger assumption of  $C^\infty$  smoothness up to the boundary.

**Theorem 2.2.** *Let  $g \in \mathbb{E}^{m, \alpha}$ ,  $m \geq 5$ , and suppose  $k$  is an infinitesimal Einstein deformation which is both Dirichlet and Neumann degenerate, so that  $L_E(k) = 0$  and (2.12) holds. Then  $k$  is pure gauge near  $\partial M$ , i.e.*

$$(2.14) \quad k = \delta^* Z \quad \text{near } \partial M,$$

with  $Z = 0$  on  $\partial M$ .

As is well-known, the operator  $E$  is not elliptic, due to its covariance under diffeomorphisms: one has  $L_E(\delta^* Y) = 0$ , for any vector field  $Y$  on  $M$ , at an Einstein metric. We will require ellipticity at several points and so need a choice of gauge to break the diffeomorphism invariance of the Einstein equations. In view of (2.8), the simplest and most natural choice is the Bianchi gauge given by  $\beta(k) = 0$  at the linearized level. (Later we will use instead a slightly different gauge, the divergence-free gauge, cf. Remark 4.6). Thus, let  $\tilde{g}$  be a fixed (background) metric in  $\mathbb{E}$ . The associated Bianchi-gauged Einstein operator is given by the  $C^\infty$  smooth map

$$(2.15) \quad \begin{aligned} \Phi_{\tilde{g}} : \text{Met}^{m, \alpha}(M) &\rightarrow S_2^{m-2, \alpha}(M), \\ \Phi(g) &= \Phi_{\tilde{g}}(g) = \text{Ric}_g - \lambda g + \delta_g^* \beta_{\tilde{g}}(g), \end{aligned}$$

where  $\beta_{\tilde{g}}(g)$  is the Bianchi operator with respect to  $\tilde{g}$ , while  $\delta^*$  is taken with respect to  $g$ . Although  $\Phi_{\tilde{g}}$  is defined for all  $g \in \text{Met}(M)$ , we will only consider it acting on  $g$  near  $\tilde{g}$ .

The linearization of  $\Phi$  at  $\tilde{g} = g$  is given by

$$(2.16) \quad \begin{aligned} L : T_{\tilde{g}} \text{Met}(M) &\rightarrow S_2(M), \\ L(h) &= 2(D\Phi)_{\tilde{g}}(h) = \nabla^* \nabla h - 2R(h). \end{aligned}$$

The operator  $L$  is formally self-adjoint and is clearly elliptic. Comparing (2.7) and (2.15), the relation between  $L$  and the linearization  $L_E = 2E'$  of the Einstein operator  $E$  in (2.8) is given by

$$(2.17) \quad L_E = L - 2\delta^* \beta.$$

In §3, we will consider elliptic boundary value problems for the operator  $\Phi$ .

Clearly  $g \in \mathbb{E}$  if  $\Phi_{\tilde{g}}(g) = 0$  and  $\beta_{\tilde{g}}(g) = 0$ , so that  $g$  is in the Bianchi gauge with respect to  $\tilde{g}$ . Given  $\tilde{g}$ , let  $\text{Met}_C(M) = \text{Met}_C^{m, \alpha}(M)$  be the space of  $C^{m, \alpha}$  smooth Riemannian metrics on  $M$  in Bianchi gauge with respect to  $\tilde{g}$  at  $\partial M$ :

$$(2.18) \quad \text{Met}_C(M) = \{g \in \text{Met}(M) : \beta_{\tilde{g}}(g) = 0 \text{ at } \partial M\}.$$

Let

$$(2.19) \quad Z_C = \{g \in \text{Met}_C(M) : \Phi(g) = 0\}$$

be the 0-set of  $\Phi$  and let  $\mathbb{E}_C \subset Z_C$  be the subset of Einstein metrics  $g$  in  $Z_C$ .

To justify the use of  $\Phi$ , one needs to show that the opposite inclusion holds, so that  $\mathbb{E}_C = Z_C$ . This has already been done in [3] and we summarize the results here.

**Lemma 2.3.** (i). For  $g$  in  $\text{Met}^{m,\alpha}(M)$ , one has

$$(2.20) \quad T_g \text{Met}^{m-2,\alpha}(M) \simeq S_2^{m-2,\alpha}(M) = \text{Ker} \delta \oplus \text{Im} \delta^*,$$

where  $\delta^*$  acts on  $\chi_1^{m-1,\alpha}$ , the space of  $C^{m-1,\alpha}$  vector fields on  $M$  which vanish on  $\partial M$ .

(ii). For  $\tilde{g} \in \mathbb{E}^{m,\alpha}$  and  $g$  in  $\text{Met}^{m,\alpha}(M)$  close to  $\tilde{g}$ , one has

$$(2.21) \quad T_g \text{Met}^{m-2,\alpha}(M) \simeq S_2^{m-2,\alpha}(M) = \text{Ker} \beta \oplus \text{Im} \delta^*,$$

where  $\beta$  is the Bianchi operator with respect to  $\tilde{g}$ . If  $g \in \mathbb{E}^{m,\alpha}$ , then (2.21) holds with  $m$  in place of  $m-2$ .

(iii). Any metric  $g \in Z_C$  near  $\tilde{g} \in \mathbb{E}^{m,\alpha}$  is Einstein, and in Bianchi gauge with respect to  $\tilde{g}$ , i.e.

$$(2.22) \quad \beta_{\tilde{g}}(g) = 0.$$

Similarly, if  $k \in \text{Met}_C(M)$  is an infinitesimal deformation of  $\tilde{g}$  in  $Z_C$ , i.e.  $L(k) = 0$ , then  $k$  is an infinitesimal Einstein deformation and  $\beta(k) = 0$ .

Lemma 2.3 implies that  $\mathbb{E}_C = Z_C$  near  $\tilde{g}$ , and at least infinitesimally  $\mathbb{E}_C$  is a local slice for the action of the diffeomorphism group  $\mathcal{D}_1$  on  $\mathbb{E}$ . In fact, it is shown in [3] that  $\mathbb{E}_C$  is a local slice for the action of  $\mathcal{D}_1$ .

The next two results may be viewed as a preliminary version of Theorem 1.1.

**Corollary 2.4.** Let  $g \in \mathbb{E}^{m,\alpha}$ ,  $m \geq 5$ , and suppose  $\kappa$  is an infinitesimal Einstein deformation of  $(M, g)$ . If  $\pi_1(M, \partial M) = 0$  and (2.12) holds, then  $\kappa$  is pure gauge on  $M$ , i.e. there exists a vector field  $Z$  on  $M$  with  $Z = 0$  on  $\partial M$  such that

$$(2.23) \quad \kappa = \delta^* Z \quad \text{on } M.$$

If  $\kappa$  is in Bianchi gauge, so that  $L(\kappa) = 0$ , then

$$\kappa = 0 \quad \text{on } M.$$

**Proof:** The hypotheses and Theorem 2.2 imply that the form  $\kappa$  on  $M$  is pure gauge near  $\partial M$ , so that (2.23) holds on a neighborhood  $\Omega$  of  $\partial M$ .

It then follows from a basically standard analytic continuation argument in the interior of  $M$  that the vector field  $Z$  may be extended so that (2.23) holds on all of  $M$ , cf. [9, §VI.6.3] for instance. A detailed proof of this is also given in [3, Lemma 2.6]. This analytic continuation argument requires the topological hypothesis (1.2) to obtain a well-defined (single-valued) vector field  $Z$  on  $M$ . Moreover, since  $\partial M$  is connected, the condition  $Z = 0$  on  $\partial M$  remains valid in the analytic continuation.

For the second statement, if in addition  $\beta(\kappa) = 0$ , then  $\beta \delta^* Z = 0$  on  $M$  with  $Z = 0$  on  $\partial M$ . It then follows from Lemma 2.3 that  $Z = 0$  on  $M$  and hence  $\kappa = 0$  on  $M$ , as claimed. ■

**Proposition 2.5.** Let  $g \in \mathbb{E}^{m,\alpha}$ ,  $m \geq 5$ , and suppose  $X$  is a Killing field on  $(\partial M, \gamma)$  such that

$$(2.24) \quad (\mathcal{L}_X A)^T = 0 \quad \text{at } \partial M,$$

If  $\pi_1(M, \partial M) = 0$ , then  $X$  extends to a Killing field on  $(M, g)$ .

**Proof:** Since  $\gamma \in \text{Met}^{m,\alpha}(\partial M)$ , the Killing field  $X$  is  $C^{m+1,\alpha}$  smooth on  $\partial M$ . By Lemma 2.3, the operator  $\beta \delta^*$  has no kernel on  $\chi_1$ . Since this operator has (Fredholm) index 0, it follows that  $X$  may be uniquely extended to a vector field  $X$  on  $M$  so that

$$(2.25) \quad \beta \delta^* X = 0 \quad \text{on } M.$$

Since  $g \in \mathbb{E}^{m,\alpha}$ , the solution  $X$  is then  $C^{m+1,\alpha}$  up to  $\partial M$ . Hence the form  $\kappa = \delta^* X$  is  $C^{m-1,\alpha}$  up to  $\partial M$  and is an infinitesimal Einstein deformation in Bianchi gauge, i.e.  $L(\kappa) = L_E(\kappa) = 0$  with  $\beta(\kappa) = 0$ . Note that by construction,  $\kappa^T = 0$  at  $\partial M$ .

Next, note that

$$(2.26) \quad \mathcal{L}_X A = 2A'_\kappa.$$

Namely, since  $\kappa = \frac{1}{2}\mathcal{L}_X g$ , as in (2.11) one has  $A'_\kappa = \frac{1}{4}\mathcal{L}_N \mathcal{L}_X g + \frac{1}{2}\mathcal{L}_{N'} g = \frac{1}{2}\mathcal{L}_X A + \frac{1}{4}\mathcal{L}_{[N,X]} g + \frac{1}{2}\mathcal{L}_{N'} g$ . It is easy to verify that  $[N, X] = -2N'$ , which gives (2.26).

The form  $\kappa$  is thus an infinitesimal Einstein deformation in Bianchi gauge and satisfies (2.12). Hence by Corollary 2.4,  $\kappa = \delta^* X = 0$  on  $M$ , which implies that  $X$  is a Killing field on  $(M, g)$ . ■

We note that if, in place of the condition  $\pi_1(M, \partial M) = 0$ , one assumes that  $(M, g)$  is embedded as a domain in a complete, simply connected Einstein manifold  $(\hat{M}, \hat{g})$ , then essentially the same analytic continuation argument (cf. [9, §VI.6.4]) again implies that  $X$  extends uniquely to a Killing field on all of  $\hat{M}$ , which proves Proposition 2.5 in this case also.

### 3. ELLIPTIC BOUNDARY VALUE PROBLEMS FOR THE EINSTEIN EQUATIONS

In this section, we consider elliptic boundary value problems for the Bianchi-gauged Einstein operator  $\Phi$  in (2.15) and the Fredholm properties of the Dirichlet boundary map  $\Pi_D$  in (2.6).

Recall that the kernel of the linearized operator  $L$  in (2.16) forms the tangent space  $T_g Z_C$  ( $g = \tilde{g}$  here) and by Lemma 2.3,

$$(3.1) \quad T_g Z_C = T_g \mathbb{E}_C,$$

so that the kernel also represents the space of (non-trivial) infinitesimal Einstein deformations in Bianchi gauge. The natural Dirichlet-type boundary conditions for  $\Phi$  are

$$(3.2) \quad \beta_{\tilde{g}}(g) = 0, \quad g^T = \gamma \quad \text{at } \partial M.$$

However, contrary to first impressions, the operator  $\Phi$  with boundary conditions (3.2) does not form a well-defined elliptic boundary value problem (for  $g$  near  $\tilde{g}$ ). This is due to the well-known constraint equations, induced by the Gauss and Gauss-Codazzi equations on  $\partial M$ :

$$(3.3) \quad \delta(A - H\gamma) = -Ric_g(N, \cdot) = 0,$$

$$(3.4) \quad |A|^2 - H^2 + s_\gamma = s_g - 2Ric_g(N, N) = (n-1)\lambda.$$

Here  $H$  is the mean curvature of  $\partial M$  in  $M$ , while  $s$  denotes the scalar curvature.

As will be seen in §4, the momentum or vector constraint (3.3) is an important issue in the study of the isometry extension or rigidity results discussed in the Introduction. On the other hand, the Hamiltonian or scalar constraint (3.4) is important in understanding the Fredholm properties of the boundary map  $\Pi_D$  in (2.6). Thus for  $g \in \mathbb{E}^{m,\alpha}$ , one has  $A \in S_2^{m-1,\alpha}(\partial M)$  so that (3.4) implies that  $s_\gamma \in C^{m-1,\alpha}(\partial M)$ . However, the space of metrics  $\gamma \in Met^{m,\alpha}(\partial M)$  for which  $s_\gamma \in C^{m-1,\alpha}(\partial M)$  is of infinite codimension in  $Met^{m,\alpha}(\partial M)$ . It follows that the linearization of the boundary map  $\Pi_D$  has infinite dimensional cokernel, at least when  $m < \infty$ , and so  $\Pi_D$  is never Fredholm. Hence, the boundary conditions (3.2) for the operator  $\Phi$  are not elliptic.

**Remark 3.1.** It is worthwhile to understand situations where the linearization  $D\Pi_D$  has infinite dimensional kernel and cokernel, even in the  $C^\infty$  case. Let

$$(3.5) \quad K = K_g = Ker D_g \Pi_D.$$

Via the slice representation  $Z_C = \mathbb{E}_C \subset \mathbb{E}$  at  $\tilde{g} = g$ ,  $K$  consists of forms  $\kappa$  such that

$$(3.6) \quad L(\kappa) = 0 \text{ and } \beta_g(\kappa) = 0 \text{ on } M, \text{ with } \kappa^T = 0 \text{ on } \partial M.$$

Consider then the intersection  $K \cap \text{Im} \delta^*$ . Let  $Y$  be a vector field at  $\partial M$  (not necessarily tangent to  $\partial M$ ) and extend  $Y$  to a vector field on  $M$  to be the unique solution to the equation  $\beta(\delta^* Y) = 0$  with the given boundary value, cf. Lemma 2.3. Then  $L(\delta^* Y) = 0$  and the boundary condition  $k^T = (\delta^* Y)^T = 0$  is equivalent to the equation

$$(3.7) \quad (\delta^* Y^T)^T + \langle Y, N \rangle A = 0 \quad \text{at } \partial M.$$

In particular if  $\delta_T^*$  is the restriction of  $\delta^*$  to vector fields tangent to  $\partial M$  at  $\partial M$ , then  $K \cap \text{Im} \delta_T^*$  is isomorphic to the space of Killing fields on  $(\partial M, \gamma)$ .

On the other hand, if  $\partial M$  is totally geodesic on some open set  $U \subset \partial M$ , i.e.  $A = 0$  on  $U$ , then the system (3.7) has solutions of the form  $Y = fN$ , for *any*  $f$  with  $\text{supp } f \subset U$ , so that  $K \cap \text{Im} \delta^*$  is infinite dimensional. Such vector fields  $Y$  are infinitesimal isometries *at* (as opposed to *on*)  $\partial M$ , in that they preserve the metric  $\gamma$  on  $\partial M$  to first order. Of course in general such  $Y$  do not extend to a Killing field on  $(M, g)$ ; see also Remark 4.3 for further discussion and examples. This behavior is classically very well-known in the context of surfaces embedded in  $\mathbb{R}^3$ , cf. [18], [7].

A similar phenomenon holds for the cokernel. Thus, suppose  $(\partial M, \gamma)$  is totally geodesic in a domain  $U \subset \partial M$ . Consider the linearization  $s'_\gamma(h)$ , for  $h \in \text{Im}(D\Pi_D)$ . By differentiating the scalar constraint (3.4) in the direction  $h$ , one sees that  $s'_\gamma(h) = 0$  on  $U$ , for any such  $h$ . It follows that  $\text{Im} D\Pi_D$  has infinite codimension, even in the  $C^\infty$  case, in such situations. The same argument and conclusion holds if  $A = 0$  at just one point in  $\partial M$ .

Very little seems to be understood in characterizing the situations where  $K$  is finite dimensional or  $K = 0$ . Again, this is the case even in the classical setting of closed surfaces embedded in  $\mathbb{R}^3$ .

The discussion above implies there is no natural elliptic boundary value problem for the Einstein equations associated with Dirichlet boundary values. To obtain an elliptic problem, one needs to add either gauge-dependent terms or terms depending on the extrinsic geometry of  $\partial M$  in  $(M, g)$ . To maintain a determined boundary value problem, one then has to subtract part of the intrinsic Dirichlet boundary data on  $\partial M$ .

There are several ways to carry this out in practice, but we will concentrate on the following situation. First, ellipticity of the Bianchi-gauged Einstein operator  $\Phi = \Phi_{\tilde{g}}$  with respect to given boundary conditions - near a given solution - depends only on the linearized operator, so we assume  $g = \tilde{g}$  is Einstein and study the linearized operator  $L$  from (2.16) at  $(M, g)$ . As usual, let  $\gamma$  be the induced metric on  $\partial M$ .

Let  $B$  be a  $C^{m, \alpha}$  symmetric bilinear form on  $\partial M$  such that

$$(3.8) \quad \tau_B = B - (\text{tr}_\gamma B) \gamma < 0,$$

is negative definite; all the statements to follow hold equally well if  $\tau_B$  is positive definite. This condition is equivalent to the statement that the sum of any  $(n-1)$ -eigenvalues of  $B$  with respect to  $\gamma$  is positive. For the choice  $B = A$ , the 2<sup>nd</sup> fundamental form, this is just the statement  $\partial M$  is  $(n-1)$ -convex in  $(M, g)$ , cf. (3.23) below.

In place of prescribing the boundary metric  $g^T$  or its linearization  $h^T$  on  $\partial M$ , only  $h^T$  modulo  $B$  will be prescribed. Thus, let  $\pi_B : T_\gamma \text{Met}^{m, \alpha}(\partial M) \rightarrow S_2^{m, \alpha}(\partial M)/B$ , be the natural projection and set  $\pi_B(h) = [h^T]_B$ . In place of the second equation in (3.2), we impose

$$(3.9) \quad [h^T]_B = h_1.$$

For example, when  $B$  equals the boundary metric  $\gamma$ , one is prescribing the trace-free part of  $h^T$ , i.e. the tangent space of conformal classes on  $\partial M$ . Another natural choice is  $B = A$ , the 2<sup>nd</sup> fundamental form of  $\partial M$ . In this case, for regularity purposes, one must work instead with a smooth approximation to  $A$ , since  $A \in S_2^{m-1, \alpha}(\partial M)$ , or with a  $C^\infty$  background  $(M, g)$ .

The simplest gauge-dependent term one can add to (3.2) is the equation  $h(N, N) = h_{00}$ , where  $N$  is the unit normal with respect to  $g$ , while the simplest extrinsic geometric scalar is the linearization



$H'_h$  of the mean curvature of  $\partial M$  in  $(M, g)$  in the direction  $h$ . As shown in [3], ellipticity holds for either of these boundary conditions. We will use a slightly more general result, whose proof is a simple modification of the proof in [3].

**Proposition 3.2.** *Suppose  $B \in S_2^{m, \alpha}$  satisfies (3.8) and suppose  $\sigma$  is any positive definite form in  $S_2^{m, \alpha}(\partial M)$ . Then the Bianchi-gauged linearized Einstein operator  $L$  in (2.16) with boundary conditions*

$$(3.10) \quad \beta(h) = 0, \quad [h^T]_B = h_1, \quad \langle A'_h, \sigma \rangle = \text{tr}_\sigma A'_h = h_2 \quad \text{at } \partial M,$$

*is an elliptic boundary value problem of Fredholm index 0.*

**Proof:** The leading order symbol of  $L = D\Phi$  is given by

$$(3.11) \quad \sigma(L) = -|\xi|^2 I,$$

where  $I$  is the  $N \times N$  identity matrix, with  $N = (n+2)(n+1)/2$  the dimension of the space of symmetric bilinear forms on  $\mathbb{R}^{n+1}$ . In the following, the subscript 0 represents the direction normal to  $\partial M$  in  $M$ , and Latin indices run from 1 to  $n$ . The positive roots of (3.11) are  $i|\xi|$ , with multiplicity  $N$ .

Writing  $\xi = (z, \xi_i)$ , the symbols of the leading order terms in the boundary operators in (3.10) are given by:

$$\begin{aligned} -2izh_{0k} - 2i \sum \xi_j h_{jk} + i\xi_k \text{tr} h &= 0, \\ -2izh_{00} - 2i \sum \xi_k h_{0k} + iz \text{tr} h &= 0, \\ h^T &= h_1 \mod B, \\ \text{tr}_\sigma A'_h &= h_2, \end{aligned}$$

where  $h$  is an  $(n+1) \times (n+1)$  matrix. Then ellipticity requires that the operator defined by the boundary symbols above has trivial kernel when  $z$  is set to the root  $i|\xi|$ . Carrying this out then gives the system

$$(3.12) \quad 2|\xi|h_{0k} - 2i \sum \xi_j h_{jk} + i\xi_k \text{tr} h = 0,$$

$$(3.13) \quad 2|\xi|h_{00} - 2i \sum \xi_k h_{0k} - |\xi| \text{tr} h = 0,$$

$$(3.14) \quad h_{kl} = \phi b_{kk} \delta_{kl},$$

$$(3.15) \quad \text{tr}_\sigma A'_h = 0,$$

where without loss of generality we assume  $B$  is diagonal, with entries  $b_{kk}$ , and  $\phi$  is an undetermined function.

Multiplying (3.12) by  $i\xi_k$  and summing gives

$$2|\xi|i \sum \xi_k h_{0k} = 2i^2 \xi_k^2 h_{kk} - i^2 \xi_k^2 \text{tr} h.$$

Substituting (3.13) on the term on the left above then gives

$$2|\xi|^2 h_{00} - |\xi|^2 \text{tr} h = -2 \sum \xi_k^2 h_{kk} + |\xi|^2 \text{tr} h,$$

so that

$$|\xi|^2 h_{00} - |\xi|^2 \text{tr} h = - \sum \xi_k^2 h_{kk} = -\phi \langle B(\xi), \xi \rangle.$$

Using the fact that  $\sum h_{kk} = \text{tr} h - h_{00}$ , this is equivalent to

$$\phi \langle B(\xi), \xi \rangle = \phi |\xi|^2 \text{tr} B.$$

Since  $\tau_B = B - (\text{tr} B)\gamma$  is assumed to be definite, it follows that  $\phi = 0$  and hence  $h^T = 0$ .

Next, a simple computation from (2.11) shows that to leading order,  $tr_\sigma A'_h = tr_\sigma(\nabla_N h - 2\delta^*(h(N)^T))$ , which has symbol  $iz\sigma^{ij}h_{ij} - 2i\sigma^{ij}\xi_i h_{0j}$ . Setting this to 0 at the root  $z = i|\xi|$  and using the fact that  $h^T = 0$  gives

$$(3.16) \quad \sigma^{ij}\xi_i h_{0j} = 0.$$

Now (3.12) and  $h^T = 0$  gives  $2|\xi|h_{0j} + i\xi_j h_{00} = 0$ . Multiplying the first term by  $\sigma^{ij}\xi_i$  and summing over  $i, j$  gives 0 by (3.16), and hence  $\sigma^{ij}\xi_i \xi_j h_{00} = 0$ . Since  $\sigma > 0$ , it follows that  $h_{00} = 0$  and hence by (3.12) again,  $h_{0k} = 0$  for all  $k$ . This gives  $h = 0$ , and hence the boundary data (3.10) are elliptic.

To prove the operator  $L$  with boundary data (3.10) is of Fredholm index 0, one may continuously deform the boundary data through elliptic boundary values to self-adjoint boundary data, which clearly has index 0. This is done in detail for the case  $\sigma = \gamma$  in [3] and the proof for general  $\sigma > 0$  is identical. Thus we refer to [3] for details as needed. The result then follows from the homotopy invariance of the index. ■

Given  $\tilde{g} \in \mathbb{E}^{m,\alpha}$ , and  $B$  as in (3.8), let  $Met_B^{m,\alpha}(\partial M) = Met^{m,\alpha}(\partial M)/B$  be the space of equivalence classes of  $C^{m,\alpha}$  metrics on  $\partial M$  (mod  $B$ ), with natural projection or quotient map

$$\pi_B : Met^{m,\alpha}(\partial M) \rightarrow Met_B^{m,\alpha}(\partial M).$$

It follows from Proposition 3.2 and Lemma 2.3 that the map

$$(3.17) \quad \tilde{\Pi}_{B,\sigma} : \mathbb{E}_C \rightarrow Met_B^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M),$$

$$\tilde{\Pi}_{B,\sigma}(g) = ([g^T]_B, tr_\sigma A),$$

is Fredholm, of index 0, for  $g$  near  $\tilde{g}$ .

In analogy to (3.5), let

$$(3.18) \quad \tilde{K}_{B,\sigma} = Ker D\tilde{\Pi}_{B,\sigma},$$

where the derivative is taken at  $g = \tilde{g}$ . In contrast to  $K$  in (3.5),  $\tilde{K}_{B,\sigma}$  is always finite dimensional. One might call an Einstein metric  $g \in \mathbb{E}$  *non-degenerate* (or  $(B, \sigma)$ -nondegenerate) if

$$(3.19) \quad \tilde{K}_{B,\sigma} = 0,$$

for some  $B, \sigma$ . Thus,  $g$  is non-degenerate if and only if  $g$  is a regular point of the boundary map  $\tilde{\Pi}_{B,\sigma}$  in which case  $\tilde{\Pi}_{B,\sigma}$  is a local diffeomorphism near  $g$ .

**Remark 3.3.** It is worth pointing out that if  $(M, g)$  is strongly non-degenerate, in the sense that  $K = 0$  in (3.5), then Theorem 1.1 is easy to prove and holds without the assumptions on  $H$  or on  $\pi_1(M, \partial M)$ . To see this, let  $\phi_s$  be a local curve of  $C^{m+1,\alpha}$  diffeomorphisms of  $\bar{M}$  with  $\phi_0 = id$  such that  $\frac{d}{ds}\phi_s|_{s=0} = X$ . If  $X$  is a Killing field on  $(\partial M, \gamma)$ , then

$$\phi_s^* \gamma = \gamma + O(s^2).$$

The curve  $g_s = \phi_s^* g$  is a smooth curve in  $\mathbb{E}$ , and by construction, one has  $[h] = [\frac{dg_s}{ds}] \in Ker D\Pi_D$ , for  $\Pi_D$  as in (2.6). One may then alter the diffeomorphisms  $\phi_s$  by composition with diffeomorphisms  $\psi_s \in \mathcal{D}_1^{m+1,\alpha}$  if necessary, so that  $\kappa = \frac{d\psi_s^*(g_s)}{ds} \in K_g$ , where  $K = K_g$  is the kernel in (3.5) and  $[h] = [\kappa]$ . Thus

$$\kappa = \delta^* X',$$

where  $X' = \frac{d(\phi_s \circ \psi_s)}{ds}$  is  $C^{m+1,\alpha}$  smooth up to  $\bar{M}$ . Note that  $X' = X$  at  $\partial M$ . If  $K_g = 0$ , then this gives

$$\delta^* X' = 0 \text{ on } M,$$

so that  $X'$  is a Killing field on  $(M, g)$ . Thus, any Killing field on  $(\partial M, \gamma)$  extends to a Killing field on  $(M, g)$ , as claimed. The same result and proof hold in general, for any infinitesimal Einstein deformation preserving the boundary metric  $(\partial M, \gamma)$ .

It follows that if this general isometry extension property fails, then the Dirichlet boundary map  $\Pi_D$  in (3.5) is necessarily degenerate.

**Remark 3.4.** Although currently the cokernel of  $D\Pi_D$  remains hard to understand, cf. Remark 3.1, it is not difficult to describe the cokernel of  $D\tilde{\Pi}_{B,\sigma}$ . For simplicity, set  $(B, \sigma) = (\gamma, \gamma)$  and let  $\tilde{\Pi}_{\gamma,\gamma} = \tilde{\Pi}_H$ . Then define

$$(3.20) \quad \tilde{C} = \{((\mathcal{L}_N \kappa)_\gamma^T, N(H'_\kappa)) : \kappa \in \tilde{K}_{\gamma,\gamma}\},$$

so that  $\tilde{C}$  represents Neumann-type data associated with the Dirichlet data in (3.9).

Note that  $\tilde{C} \subset S_\gamma^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M)$ , where  $S_\gamma^{m,\alpha}(\partial M) = T_g \text{Met}_\gamma^{m,\alpha}(\partial M) \simeq S^{m,\alpha}(\partial M)/\langle \gamma \rangle$ . Namely, for  $\kappa \in \tilde{K}_{\gamma,\gamma}$ , one has  $L(\kappa) = 0$  on  $M$  together with the elliptic boundary conditions  $\beta(\kappa) = 0$ ,  $\kappa_\gamma^T = 0$ , and  $H'_\kappa = 0$  on  $\partial M$ . Since  $g$  is  $C^{m,\alpha}$  up to  $\partial M$ , elliptic boundary regularity applied to this system gives  $\kappa \in C^{m+1,\alpha}$  (cf. [8, 12]) so that  $\mathcal{L}_N \kappa \in S_2^{m,\alpha}(\partial M)$  and  $N(H'_\kappa) \in C^{m-1,\alpha}(\partial M)$ .

It is then not difficult to prove (although we will not give the proof here) that the space  $\tilde{C}$  is a slice for  $\text{Coker } D\tilde{\Pi}_H$  in  $S_\gamma^{m,\alpha}(\partial M) \times C^{m,\alpha}(\partial M)$ , so that

$$(3.21) \quad S_\gamma^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M) = \text{Im } D\tilde{\Pi}_H \oplus \tilde{C}.$$

By restricting to the first factor, it follows immediately from (3.21) that

$$(3.22) \quad S_\gamma^{m,\alpha}(\partial M) = \text{Im } D\Pi_0 \oplus \tilde{S},$$

where  $\tilde{S} = \{(\mathcal{L}_N \kappa)_\gamma^T : \kappa \in \tilde{K}_H\}$  and  $\Pi_0$  is defined by  $\Pi_0 = \pi_\gamma \circ \Pi_D$ .

One may use the diffeomorphism group to pass from the space  $\mathbb{E}_C$  of Bianchi-gauged Einstein metrics to the full space  $\mathbb{E}$ , thus passing from  $\tilde{\Pi}_H$  to the more natural Dirichlet boundary map  $\Pi_D$ . In more detail, the image  $\mathcal{V} = D\Pi_D(\mathbb{E}_C) \subset T\text{Met}^{m,\alpha}(\partial M)$  projects onto a space of finite codimension in  $S_\gamma^{m,\alpha}(\partial M)$  by (3.22). The full image  $D\Pi_D(\mathbb{E})$  then consists of the span  $\langle \mathcal{V}, \text{Im } \delta^* \rangle$ , where  $\delta^*$  acts on all vector fields at  $\partial M$ , not necessarily tangent to  $\partial M$ . It is an interesting question to understand when the closure of this space is of finite codimension in  $T\text{Met}^{m,\alpha}(\partial M)$ . This corresponds roughly to  $\Pi_D$  being Fredholm.

One situation where this occurs is the following. Define  $\partial M \subset M$  to be  $p$ -convex if the sum of any  $p$  eigenvalues of the second fundamental form  $A$  of  $\partial M$  in  $(M, g)$  is positive, cf. also [17] for example. Thus,  $\partial M$  is 1-convex if  $A > 0$  is positive definite, while  $\partial M$  is  $n$ -convex if  $H > 0$ . It is easy to see that  $A$  is  $(n-1)$ -convex if and only if the form  $H\gamma - A$  is positive definite,

$$(3.23) \quad H\gamma - A > 0.$$

This condition is equivalent to the local convexity of  $\partial M$  in  $(M, g)$  when  $n = 2$ , but becomes progressively weaker in higher dimensions.

**Proposition 3.5.** *If  $\partial M$  is  $(n-1)$ -convex, so that (3.23) holds, then the space*

$$\mathcal{V} = \overline{\text{Im } D\Pi_D},$$

*is of finite codimension in  $S_2^{m,\alpha}(\partial M)$ , where the closure is taken in the  $C^{m-1,\alpha}$  topology.*

**Proof:** Recall from Proposition 3.2 that the operator  $L$  in (2.16) with boundary data

$$(3.24) \quad \beta(h) = 0, \quad [h^T]_B = h_1, \quad \text{tr}_\sigma A'_h = h_2,$$

is elliptic, of Fredholm index 0, provided  $\sigma$  is positive (or negative) definite and provided  $\tau_B = B - (\text{tr}_\gamma B)\gamma \in S_2^{m,\alpha}(\partial M)$ , is also negative definite. For  $B = A$ , by (3.23) one has the required

definiteness, but there is a loss of one derivative in that  $\tau \in S_2^{m-1,\alpha}(\partial M)$ . Thus, let  $A_\varepsilon$  be a  $(C^\infty)$  smoothing of  $A$ ,  $\varepsilon$ -close to  $A$  in the  $C^{m-1,\alpha}$  topology. Then the system  $L(h) = 0$  with boundary data

$$(3.25) \quad \beta(h) = 0, \quad [h^T]_{A_\varepsilon} = h_1, \quad \text{tr}_\sigma A'_h = h_2,$$

is elliptic, of Fredholm index 0. The kernel and cokernel are of finite and equal dimensions.

Let  $\pi_{A_\varepsilon}$  denote the projection onto  $S_2^{m,\alpha}(\partial M)/\langle A_\varepsilon \rangle = S_\varepsilon^{m,\alpha}(\partial M)$ . Then the image of  $\pi_{A_\varepsilon} \circ D\Pi_D$  is of finite codimension in  $S_{A_\varepsilon}^{m,\alpha}(\partial M)$ . The fiber  $(\pi_{A_\varepsilon})^{-1}(0)$  consists of symmetric forms of the form  $fA_\varepsilon$ . Note that the forms  $fA$  are in  $\text{Im} D\Pi_D$ , when  $\Pi_D$  is extended to the domain  $\mathbb{E}^{m-1,\alpha}(M)$ , in that

$$fA = \delta^*(fN) \quad \text{at } \partial M,$$

where  $\delta^*(fN)$  is extended to  $M$  to be in Bianchi gauge. Since the forms  $fA$  are  $C^\varepsilon$ -close to  $fA_\varepsilon$  in the  $C^{m-1,\alpha}$  topology, when  $|f|_{C^{m-1,\alpha}} \leq C$ , it follows (by letting  $\varepsilon \rightarrow 0$ ) that the closure of  $\text{Im} D\Pi_D$  is of finite codimension in  $S_2^{m,\alpha}(\partial M)$ . ■

**Remark 3.6.** Consider hyperbolic 3-space  $\mathbb{H}^3(-1)$  divided by translation along a geodesic, giving a hyperbolic metric  $g_{-1}$  on  $D^2 \times S^1$ . The metric  $g_{-1}$  has the simple form

$$g_{-1} = dr^2 + \sinh^2 r (d\theta_1)^2 + \cosh^2 r (d\theta_2)^2.$$

As in the example discussed in the Introduction, let  $\sigma$  be any smooth embedded closed curve in the hyperbolic plane  $D^2 = \mathbb{H}^2(-1)$  surrounding the origin and let  $D$  be the disc bounded by  $\sigma$ . Let  $M = \pi^{-1}(D^2) \simeq D^2 \times S^1$  with  $\partial M = \pi^{-1}(\sigma)$ , so that  $M$  is a solid torus with boundary a flat torus  $T^2$ .

It is easy to see that  $\partial M$  is convex in  $M$  whenever  $\sigma$  is convex in  $\mathbb{H}^2(-1)$ . However the flat torus boundary has two Killing fields, only one of which (namely the vertical field tangent to  $\theta_2$ ) extends to a Killing field on  $M$  whenever the geodesic or mean curvature of  $\sigma$  in  $\mathbb{H}^2(-1)$  is non-constant. Thus, isometry extension fails, even though  $\partial M$  is strictly convex - in contrast to the case of rigidity of convex surfaces in  $\mathbb{R}^3$ , cf. [18].

#### 4. ISOMETRY EXTENSION AND THE DIVERGENCE CONSTRAINT

By Proposition 2.5, the basic issue for the isometry extension property is to understand when a Killing field on  $(\partial M, \gamma)$  preserves the 2<sup>nd</sup> fundamental form  $A$  of  $\partial M$  in  $M$ . We begin with the following identity on  $(\partial M, \gamma)$ , which holds on any closed oriented Riemannian manifold.

**Proposition 4.1.** *Let  $X$  be a Killing field on  $(\partial M, \gamma)$ . Suppose  $\tau$  is a divergence-free symmetric bilinear form on  $(\partial M, \gamma)$ . Then*

$$(4.1) \quad \int_{\partial M} \langle \mathcal{L}_X \tau, h \rangle dV_\gamma = -2 \int_{\partial M} \langle \delta' \tau, X \rangle dV_\gamma,$$

where  $\mathcal{L}_X$  is the Lie derivative with respect to  $X$  and  $\delta' = \frac{d}{ds} \delta_{\gamma+sh}$  is the variation of the divergence on  $(\partial M, \gamma)$  in the direction  $h \in S_2(\partial M)$ .

**Proof:** Since the flow of  $X$  preserves  $\gamma$ , one has

$$(4.2) \quad \int_{\partial M} \langle \mathcal{L}_X \tau, h \rangle dV_\gamma = - \int_{\partial M} \langle \tau, \mathcal{L}_X h \rangle dV_\gamma.$$

Next, setting  $\gamma_s = \gamma + sh$ , the divergence theorem applied to the 1-form  $\tau(X)$  on  $\partial M$  gives

$$(4.3) \quad 0 = \int_{\partial M} \delta_{\gamma_s}(\tau(X)) dV_{\gamma_s} = \int_{\partial M} \langle \delta_{\gamma_s} \tau, X \rangle dV_{\gamma_s} - \frac{1}{2} \int_{\partial M} \langle \tau, \mathcal{L}_X \gamma_s \rangle dV_{\gamma_s},$$

where the second equality is a simple computation from the definitions; the inner products are with respect to  $\gamma_s$ . Taking the derivative with respect to  $s$  at  $s = 0$  and using the facts that  $X$  is a Killing field on  $\partial M$  and  $\delta\tau = 0$ , it follows that

$$\int_{\partial M} \langle \delta'\tau, X \rangle dV - \frac{1}{2} \int_{\partial M} \langle \tau, \mathcal{L}_X h \rangle dV = 0.$$

Combining this with (4.2) then gives (4.1). ■

We now examine the right side of (4.1) in connection with the divergence constraint (3.3); of course (3.3) implies that the form

$$\tau_A \equiv \tau = A - H\gamma,$$

cf. (3.8), is divergence-free on  $\partial M$ .

We first discuss the general perspective. As discussed in §2, one may view the pair  $(\gamma, A)$  as Cauchy data for the Einstein equations (2.3) at  $\partial M$ . The data  $(\gamma, A)$  are then formally freely specifiable subject to the constraints (3.3)-(3.4). Let  $\mathcal{T}$  be the space of pairs  $(\gamma, \tau)$  with  $\tau$  divergence-free with respect to  $\gamma$ ; here  $\gamma \in Met^{m,\alpha}(\partial M)$ ,  $\tau \in S_2^{m-1,\alpha}(\partial M)$ . One has a natural projection onto the first factor

$$(4.4) \quad \pi : \mathcal{T} \rightarrow Met^{m,\alpha}(\partial M),$$

Let also  $\mathcal{F} \subset \mathcal{T}$  be the subset of pairs satisfying the scalar constraint equation (3.4). When expressed in terms of  $\tau = A - H\gamma$ , (3.4) is equivalent to

$$|\tau|^2 - \frac{1}{n-1} (tr\tau)^2 + s_\gamma = (n-1)\lambda.$$

Pairs  $(\gamma, \tau) \in \mathcal{F}$  determine formal solutions of the Einstein equations near  $\partial M$ . More precisely, let  $(t, x^i)$  be geodesic boundary coordinates for  $(M, g)$ , so that by the Gauss Lemma, the metric  $g$  has the form

$$(4.5) \quad g = dt^2 + g_t,$$

where  $t(x) = dist_g(x, \partial M)$  and  $g_t$  is the induced metric on the level set  $S(t)$  of  $t$ . Pulling back by the flow lines of  $\nabla t$ ,  $g_t$  may be viewed as a curve of metrics on  $\partial M$ , and one may formally expand  $g_t$  in its Taylor series:

$$(4.6) \quad g_t \sim \gamma - tA - \frac{1}{2}t^2\dot{A} + \dots,$$

where  $\dot{A} = \nabla_N A = -\nabla_T A$ ,  $T = \nabla t = -N$ . As noted above, the terms  $(\gamma, A)$  are freely specifiable, subject to the constraints (3.3)-(3.4). All the higher order terms in the expansion (4.6) are then determined by  $\gamma$  and  $A$ . To see this, one first uses the standard Riccati equation

$$(4.7) \quad \nabla_T A + A^2 + R_T = 0,$$

where  $R_T(X, Y) = \langle R(X, T)T, Y \rangle$ , cf. [13]. A standard formula gives  $\nabla_T A = \mathcal{L}_T A - 2A^2$ . Also, by the Gauss equation, the curvature term  $R_T$  may be expressed as

$$R_T = Ric^T - Ric_{int} + HA - A^2,$$

where  $H = trA$ ,  $Ric_{int}$  is the intrinsic Ricci curvature of  $S(t)$  and  $Ric^T$  is the tangential part (tangent to  $S(t)$ ) of the ambient Ricci curvature. Substituting in (4.7) gives

$$(4.8) \quad \ddot{g} = -2Ric^T + 2Ric_{int} + 4A^2 - 2HA.$$

For Einstein metrics satisfying (2.3), the right side of (4.8) involves only the first order  $t$ -derivatives of the metric  $g$ . Thus, repeated differentiation of (4.8) shows that all derivatives  $g_{(k)} = \mathcal{L}_T^k g$  are determined at the boundary  $M$  by the Cauchy data  $(\gamma, A)$ , so that  $(\gamma, A)$  determines the formal Taylor expansion of the curve  $g_t$  in (4.5) at  $t = 0$ .

The Cauchy-Kovalevsky theorem implies that if  $(\gamma, \tau)$  are real-analytic forms on  $\partial M$ , then the formal series (4.6) converges to  $g_t$ , so that one obtains an actual Einstein metric  $g$  as in (4.5), defined in a neighborhood of  $\partial M$ . Of course, such metrics will not in general extend to globally defined Einstein metrics on  $M$ .

Now the right side of (4.1) is closely related to the linearization of the divergence constraint. Thus, if  $(\gamma_s, \tau_s)$  is a curve in  $Met^{m,\alpha}(\partial M) \times S_2^{m-1,\alpha}(\partial M)$  with tangent vector  $(\gamma', \tau') = (h, \tau')$  at  $s = 0$ , then by the Gauss-Codazzi equation one has

$$(4.9) \quad \delta'(\tau) + \delta(\tau') = -(Ric(N, \cdot))',$$

where  $\delta'$  is defined as in (4.1). If  $(\gamma_s, \tau_s)$  is a curve in  $\mathcal{T}$ , then

$$(4.10) \quad \delta'(\tau) + \delta(\tau') = 0;$$

this is the linearized divergence constraint.

**Lemma 4.2.** *If the derivative  $D\pi$  in (4.4) is surjective at  $(\gamma, \tau)$ ,  $\tau = A - H\gamma$ , then*

$$(4.11) \quad \mathcal{L}_X A = 0 \quad \text{on } \partial M,$$

*for any Killing field  $X$  on  $(\partial M, \gamma)$ . Conversely, if (4.11) holds for all such Killing fields  $X$ , then  $D\pi$  is surjective.*

**Proof:** This result follows easily from Proposition 4.1, with  $\tau = A - H\gamma$ . Thus, (4.10) gives  $\delta'(\tau) = -\delta(\tau')$ , for the variation  $\delta'$  of  $\delta$  in any direction  $h \in T_\gamma Met(\partial M)$ , for some  $\tau'$ . Hence, (4.1) gives

$$(4.12) \quad \mathcal{F}(h) = \int_{\partial M} \langle \mathcal{L}_X \tau, h \rangle = -2 \int_{\partial M} \langle \delta(\tau'), X \rangle = 2 \int_{\partial M} \langle \tau', \delta^* X \rangle = 0,$$

since  $X$  is a Killing field on  $(\partial M, \gamma)$ . Since  $h$  is arbitrary, this implies that

$$\mathcal{L}_X \tau = 0,$$

on  $\partial M$ , and (4.11) follows by taking the trace of this equation. The same proof also gives the converse as well, using the splitting (4.13) below.  $\blacksquare$

Thus, given  $g \in \mathbb{E}$  and its corresponding 2<sup>nd</sup> fundamental form  $A$ , giving the pair  $(\gamma, A)$  at  $\partial M$ , a fundamental issue is whether  $D\pi$  is surjective at  $(\gamma, A)$ , i.e. whether the linearized divergence constraint (4.10) is solvable, for any variation  $h$  of  $\gamma$  on  $\partial M$  (or for a space of variations dense in  $S_2(\partial M)$  in the  $L^2$  norm). One cannot expect that this holds at a general pair  $(\gamma, \tau) \in \mathcal{T}$ . Namely, for any compact manifold  $\partial M$ , one has

$$(4.13) \quad \Omega^1(\partial M) = Im\delta \oplus Ker\delta^*,$$

where  $\Omega^1$  is the space of  $(C^{m-1,\alpha})$  1-forms on  $\partial M$ . Thus, solvability at  $(\gamma, \tau)$  in general requires that

$$(4.14) \quad \delta'(\tau) \in Im\delta = (Ker\delta^*)^\perp.$$

Of course  $Ker\delta^*$  is exactly the space of Killing fields on  $(\partial M, \gamma)$ , and so this space serves as a potential obstruction space.

Obviously,  $\pi$  is locally surjective when  $(\partial M, \gamma)$  has no Killing fields. On the other hand, it is easy to construct examples where  $(\partial M, \gamma)$  does have Killing fields and  $\pi$  is not locally surjective.

**Example 4.3.** Let  $(\partial M, \gamma)$  be a flat metric on the  $n$ -torus  $T^n$ ; for example  $\gamma = d\theta_1^2 + \cdots + d\theta_n^2$ . Let  $\tau = f(\theta_1)d\theta_2^2$  (for example). Then  $\delta\tau = 0$ , for any  $C^1$  function  $f(\theta_1)$ . The pair  $(\gamma, \tau)$  is in  $\mathcal{T}$ , and in fact in  $\mathcal{F} \subset \mathcal{T}$ . Letting  $X$  be the Killing field  $\partial_{\theta_1}$ , one has  $\mathcal{L}_X \tau \neq 0$  whenever  $f$  is non-constant, so that by the converse of Lemma 4.2,  $\pi$  is not locally surjective at such  $(\gamma, \tau)$ .

If  $(\gamma, \tau)$  above are real-analytic, then  $(\partial M, \gamma)$  is the boundary metric of an Einstein metric defined on a thickening  $\partial M \times I$  of  $\partial M$ . Of course in general, such thickenings will not extend to Einstein metrics on a compact manifold bounding  $\partial M$ .

To obtain examples on compact manifolds, one may use the examples of  $\mathbb{R}^2 \times S^1$ ,  $S^3$  or  $\mathbb{H}^3(-1)/\mathbb{Z}$  discussed in the Introduction and Remark 3.5. Here one has an infinite dimensional space of isometric embeddings of a flat torus in  $\mathbb{R}^2 \times S^1$ ,  $S^3$  or  $\mathbb{H}^3(-1)/\mathbb{Z}$  for which Killing fields on the boundary do not extend to Killing fields of the ambient space.

Now clearly  $D\pi$  is surjective onto  $Im D\Pi_D$ , since  $Im D\Pi_D$  consists of variations of the boundary metric determined by global variations of the Einstein metric  $g$  on  $M$  which of course satisfy (4.10). Hence if  $D\Pi_D$  is onto, or has dense range in  $S_2(\partial M)$ , then Lemma 4.2 holds, i.e. (4.11) holds; compare with Remark 3.3. On the other hand, the examples above show that whether (4.11) holds or not must depend either on global properties of  $(M, g)$  or extrinsic properties of  $\partial M \subset M$ .

Next, we place the discussion above in a broader context of rigidity issues. The boundary  $(\partial M, \gamma)$  of the Einstein manifold  $(M, g)$  is called infinitesimally (Einstein) rigid if the kernel  $K$  of  $D\Pi_D$  in (3.5) is trivial, i.e.  $K = 0$ . Thus, infinitesimal rigidity is equivalent to the injectivity of  $D\Pi_D$ . It is also equivalent to the local rigidity of  $(\partial M, \gamma)$  (i.e. the local uniqueness of an Einstein filling  $(M, g)$  up to isometry) by the manifold theorem, Theorem 2.1.

Suppose  $X$  is an infinitesimal isometry at  $(\partial M, \gamma)$ , in that  $(\delta^* X)^T = 0$  at  $\partial M$  ( $X$  is not necessarily tangent to  $\partial M$ ). Then as discussed in Remark 3.1, the deformation  $\delta^* X$  may be extended uniquely to  $M$  by choosing it to be in Bianchi gauge. Then  $\delta^* X \in K$  and infinitesimal rigidity of  $\partial M$  implies that  $k = 0$ , so that  $X$  is a Killing field on  $(M, g)$ . Rigidity in this more restricted sense will be called infinitesimal isometric rigidity. Both forms of such rigidity are of course generalizations of the isometry extension property discussed in the Introduction.

One may obtain analogs of Proposition 4.1 and Lemma 4.2 in this context via the Einstein-Hilbert action. Thus, recall that Einstein-Hilbert action with Gibbons-Hawking-York boundary term on  $M$  is

$$(4.15) \quad I(g) = I_{EH}(g) = - \int_M (s_g - 2\Lambda) dV_g - 2 \int_{\partial M} H dv_\gamma,$$

where  $\Lambda = \frac{n-1}{2}\lambda$ , cf. [9]. The 1<sup>st</sup> variation of  $I$  in the direction  $h$  is given by

$$(4.16) \quad \frac{d}{dr} I(g + rh) = \int_M \langle \hat{E}_g, h \rangle dV_g + \int_{\partial M} \langle \tau_g, h \rangle dv_\gamma,$$

where  $\hat{E}$  is the Einstein tensor,

$$(4.17) \quad \hat{E}_g = Ric_g - \frac{s}{2}g + \Lambda g,$$

and  $\tau = A - H\gamma$  is as above. Here and below, all parameter derivatives are taken at 0. Einstein metrics with  $Ric_g - \lambda g = 0$  are critical points of  $I$ , among variations vanishing on  $\partial M$ . Consider a 2-parameter family of metrics  $g_{r,s} = g + rh + sk$  where  $E_g = 0$ . Then

$$(4.18) \quad \frac{d^2}{dsdr} I(g_{r,s}) = \frac{d^2}{drds} I(g_{r,s}).$$

Computing the left side of (4.18) by taking the derivative of (4.16) in the direction  $k$  gives

$$(4.19) \quad \frac{d^2}{dsdr} I(g_{r,s}) = \int_M \langle \hat{E}'(k), h \rangle dV_g + \int_{\partial M} \langle \tau'_k + a(k^T), h^T \rangle dv_\gamma.$$

Since  $\hat{E}_g = 0$ , there are no further derivatives of the bulk integral in (4.16). Also,  $a(k) = -2\tau \circ k + \frac{1}{2}(tr_\gamma k)\tau$  arises from the variation of the metric and volume form in the direction  $k$ ; by definition  $(\tau \circ k)(V, W) = \frac{1}{2}\{\langle \tau(V), k(W) \rangle + \langle \tau(W), k(V) \rangle\}$ .

Similarly, for the right side of (4.18) one has

$$(4.20) \quad \frac{d^2}{drds} I(g_{r,s}) = \int_M \langle \hat{E}'(h), k \rangle dV_g + \int_{\partial M} \langle \tau'_h + a(h^T), k^T \rangle dv_\gamma.$$

In particular, suppose  $k_D$  is an infinitesimal Einstein deformation in the kernel  $K$  from (3.5), so that  $k_D|_{\partial M} = k^T = 0$ . If  $h \in T\mathbb{E}$  is any infinitesimal Einstein deformation, then (4.18)-(4.20) gives,

$$(4.21) \quad \int_{\partial M} \langle \tau'_{k_D}, h \rangle dv_\gamma = \int_{\partial M} \langle \tau'_h, k_D \rangle dv_\gamma = 0.$$

One thus has

$$I''(k_D, h) = 0,$$

on-shell. Note this computation recaptures (4.12) when  $k_D = \delta^* X = \frac{1}{2} \mathcal{L}_X g$ .

**Proposition 4.4.** *If  $\pi_1(M, \partial M) = 0$  and the linearization  $D\Pi_D$  has dense range in  $S_2^{m,\alpha}(\partial M)$ , then  $D\Pi_D$  is injective, so that  $K = 0$  in (3.5) and  $(\partial M, \gamma)$  is infinitesimally Einstein rigid.*

**Proof:** The proof is a simple consequence of (4.16)-(4.19) and Corollary 2.4. Thus, suppose  $k \in K$  so that  $k$  is an infinitesimal Einstein deformation with  $k^T = 0$  at  $\partial M$ . By (4.21),

$$(4.22) \quad \int_{\partial M} \langle \tau'_k, h \rangle = \int_{\partial M} \langle \tau'_h, k \rangle = 0,$$

for any  $h \in \text{Im} D\Pi_D$ . Since  $\text{Im} D\Pi_D$  is dense in  $S_2^{m,\alpha}(\partial M)$ , it follows that  $(\tau'_k)^T = 0$  on  $\partial M$ . Taking the trace, it follows that

$$k^T = 0 \quad \text{and} \quad (A'_k)^T = 0 \quad \text{on} \quad \partial M.$$

It now follows from Corollary 2.4 that  $k = \delta^* Z$  with  $Z = 0$  on  $M$ , so that  $k$  is pure gauge. This means that the equivalence class  $[k] = 0$  in  $T\mathcal{E}$ . Alternately, assuming without loss of generality that  $k$  is in the Bianchi slice  $\beta(k) = 0$ , it follows again from Corollary 2.4 that  $k = 0$ , which proves the result. ■

It is an open question whether converse holds, i.e. if the injectivity of  $D\Pi_D$  implies  $D\Pi_D$  has dense range. By the discussion in §3,  $D\Pi_D$  is never surjective onto  $S_2^{m,\alpha}(\partial M)$ , when  $m < \infty$ .

**Remark 4.5.** There are simple examples of Einstein metrics which are not infinitesimally rigid, even when  $\partial M$  is convex. Perhaps the simplest example is given by the curve of Riemannian Schwarzschild metrics  $g_m$  on  $\mathbb{R}^2 \times S^2$ , given by

$$g_m = V^{-1} dr^2 + V d\theta^2 + r^2 g_{S^2(1)},$$

where  $V = V(r) = 1 - \frac{2m}{r}$ ,  $r \geq 2m > 0$ . Smoothness at the horizon  $\{r = 2m\}$  requires that  $\theta \in [0, \beta]$  where  $\beta = 8\pi m$ , so that  $g_m$  may be rewritten in the form

$$(4.23) \quad g_m = V^{-1} dr^2 + 64\pi^2 m^2 V d\theta^2 + r^2 g_{S^2(1)},$$

where now  $\theta \in [0, 1]$ . This is a curve of complete Ricci-flat metrics, but the metrics  $g_m$  differ from each other just by rescalings and diffeomorphisms. Taking the derivative with respect to  $m$  gives an infinitesimal Einstein deformation  $\kappa$  of  $g_m$ :

$$(4.24) \quad \kappa = 2m64\pi^2 \left[1 - \frac{3m}{r}\right] d\theta^2 + \frac{2}{r} \left(1 - \frac{2m}{r}\right)^{-2} dr^2.$$

For the moment, fix  $m > 0$  and let  $M = M(R) = \{2m \leq r \leq R\}$ . The restriction of  $g_m$  to  $M$  gives a curve of Einstein metrics on the bounded domain  $D^2 \times S^2$  with boundary  $\partial M \simeq S^1 \times S^2$  and boundary metric

$$\gamma = \gamma_R = 64\pi^2 m^2 \left[1 - \frac{2m}{R}\right] d\theta^2 + R^2 g_{S^2(1)}.$$



Let  $\omega(R)$  be the ratio of the radii of the  $S^1$  and  $S^2$  factors at  $\partial M$ , so that

$$\omega(R) = \frac{64\pi^2 m^2 [1 - \frac{2m}{R}]}{R^2}.$$

Then  $\omega(R) \rightarrow 0$  as  $R \rightarrow 0$  and  $R \rightarrow \infty$ , and has a single maximum value  $64\pi^2/27$  at the critical point  $R = 3m$  where  $\kappa^T = 0$ . At this critical radius, equal to the photon radius of the Lorentzian Schwarzschild metric, the boundary metric has the form

$$\gamma = \frac{64}{3}\pi^2 m^2 d\theta^2 + 9m^2 g_{S^2(1)},$$

and a simple calculation shows that the 2<sup>nd</sup> fundamental form  $A$  is umbilic, with

$$A = \frac{1}{3\sqrt{3}m}\gamma.$$

The discussion above shows that the Einstein metric  $g_m$  is not infinitesimally rigid on the domain  $M(3m)$ ; the form  $\kappa$  in (4.24) is in  $\text{Ker } D\Pi_D$ . Proposition 4.4 implies that  $D\Pi_D$  does not have dense range on  $M(3m)$ ; in fact boundary metrics for which the mass-independent ratio  $\omega > \omega_0 = 64\pi^2/27$  are not in  $\text{Im } \Pi_D$  (at least along the Schwarzschild curve). The Dirichlet boundary map  $\Pi_D$  has a simple fold behavior near the critical radius, and so has local degree 0. It is shown in [19] that the Schwarzschild metric  $g_m$  on  $M(R)$  is stable, in that the 2<sup>nd</sup> variation of the action (4.15) is positive definite, for  $R < 3m$ , while it becomes unstable, has a negative mode or eigenvalue when  $R > 3m$ .

A detailed discussion of the physical aspects of the Schwarzschild curve is given in [19], and further examples in both four and higher dimensions are discussed in [1] and references therein.

A simple computation using (2.11) shows that on the domain  $M(3m)$

$$A'_\kappa = \frac{1}{\sqrt{3}m^2}(\theta^1)^2 - \frac{1}{3\sqrt{3}m^2}\gamma,$$

where  $\theta^1$  is the unit 1-form in the direction  $\theta$ . This shows that  $H'_\kappa = 0$  at  $\partial M$ . Hence, the form  $\kappa$  is also in the kernel of the Fredholm boundary map  $\tilde{\Pi}_{B,\gamma}$  in (3.17). This shows that the generalization of Theorem 1.1 to infinitesimal Einstein rigidity is false; the form  $\kappa$  is a non-trivial infinitesimal Einstein deformation preserving the boundary metric and mean curvature. Of course  $\kappa$  is not of the form  $\delta^*X$  for some vector field  $X$ .

**Remark 4.6.** The work above has been carried out with the operator  $E(g) = \text{Ric}_g - \lambda g$  in (2.7), and in the Bianchi gauge, since the computations are the simplest in this setting. However, the discussion following (4.16) suggests that the “physical” Einstein operator

$$\hat{E}(g) = \text{Ric}_g - \frac{s}{2}g + \Lambda g$$

in (4.17) may be more natural in certain respects. This is in fact the case, and will be used in §5.

To set the stage, note first that the analog of the Bianchi identity in this setting is  $\delta\hat{E} = 0$ . As in §2, fix any background Einstein metric  $\tilde{g}$  and consider the operator

$$(4.25) \quad \hat{\Phi}_{\tilde{g}}(g) = \text{Ric}_g - \frac{s}{2}g + \Lambda g + \delta_g^* \delta_{\tilde{g}}(g),$$

(cf. (2.15)). The linearization of  $\hat{\Phi}$  at  $g = \tilde{g}$  is

$$(4.26) \quad \hat{L}(h) = \nabla^* \nabla h - 2R(h) - (D^2 \text{tr} h + \delta \delta h)g + \Delta \text{tr} h g,$$

where  $D^2$  is the Hessian and  $\Delta = \text{tr} D^2$  the Laplacian (with respect to  $g$ ). This is more complicated than (2.16), but it is easy to see that  $\hat{L}$  is formally self-adjoint; this also follows directly from the symmetry of the 2<sup>nd</sup> derivatives in (4.18)-(4.19). Of course solutions of  $\hat{L}(h) = 0$  with  $\delta(h) = 0$  on  $M$  are infinitesimal Einstein deformations.

It is straightforward to see that the operator  $\hat{L}$  with boundary conditions

$$(4.27) \quad \delta(h) = 0, \quad [h^T]_0 = h_1, \quad H'_h = h_2 \quad \text{at } \partial M,$$

(where  $[h^T]_0 = [h^T]\gamma$ , the trace-free part of  $h^T$ ) is a well-posed elliptic boundary value problem. This follows from Proposition 3.2, using the fact that the change from Bianchi to divergence-free gauge is unique; it is also proved directly in [4]. Moreover, it is not only of Fredholm index 0, but the boundary value problem (4.27) is self-adjoint (which is not the case for  $L$  with the boundary conditions (3.10) from Proposition 3.2) cf. again [4].

All of the remaining discussion in §2 carries over immediately to this setting; the only change is that one replaces the Bianchi operator  $\beta$  by the divergence operator  $\delta$ . For instance, the analog of (2.17) is

$$(4.28) \quad L_{\hat{E}} = \hat{L} - 2\delta^*\delta.$$

Similarly, Lemma 2.3 holds in this setting, with the same proof.

## 5. PROOF OF THE MAIN RESULTS.

In this section, we prove the main results discussed in the Introduction, beginning with Theorem 1.1. As noted above, one needs to use global arguments to prove Theorem 1.1. We do this by studying global properties of the linearized operator  $\hat{L}$  from (4.26).

Consider then the elliptic boundary value problem (4.26)-(4.27):

$$(5.1) \quad \hat{L}(h) = \ell, \quad \text{on } M, \quad \delta(h) = h_0, \quad [h^T]_0 = h_1, \quad H'_h = h_2 \quad \text{on } \partial M,$$

As noted in Remark 4.6, this is a self-adjoint elliptic boundary value problem. The self-adjoint property leads to significant simplifications in the proof, which is why we use the divergence gauge and  $\hat{L}$  in place of  $L$  and the Bianchi gauge.

Let  $K$  denote the kernel, so that  $k \in K$  means

$$\hat{L}(k) = 0, \quad \delta(k) = 0, \quad [k^T]_0 = 0, \quad H'_k = 0.$$

If  $K = 0$ , then  $L$  is surjective and so the form  $\ell$  and boundary values for  $\delta(h)$ ,  $[h^T]_0$  and  $H'_h$  may be freely chosen; given arbitrary  $\ell$  and  $h_i$ ,  $0 \leq i \leq 2$ , the system (5.1) has a unique solution, (when suitable smoothness assumptions are imposed).

Now (regardless of whether  $K = 0$  or not) as in Lemma 4.2 (using (4.10) and (4.12)) one has

$$(5.2) \quad \int_{\partial M} \langle \mathcal{L}_X \tau, h^T \rangle = \int_{\partial M} (Ric(N, X))'_h.$$

We will prove that any deformation  $h^T$  of  $\gamma$  on  $\partial M$  extends to a deformation  $h$  of  $g$  on  $M$  such that the right side of (5.2) vanishes; Theorem 1.1 then follows easily.

Note first that (5.2) vanishes in pure-trace directions  $h^T = f\gamma$ . Namely, since  $X$  is Killing,  $tr(\mathcal{L}_X \tau) = -(n-1)X(H) = 0$ , by assumption. Hence,  $\langle \mathcal{L}_X \tau, f\gamma \rangle = 0$  pointwise and so the right side of (5.2) vanishes in pure-trace directions also.

By Lemma 2.3 and Remark 4.6, deformations  $h$  satisfying

$$(5.3) \quad \hat{L}(h) = 0, \quad \text{on } M, \quad \delta(h) = 0 \quad \text{on } \partial M,$$

are infinitesimal Einstein deformations in divergence-free gauge on  $M$  and hence, at  $\partial M$ ,

$$(Ric(N, X))'_h = 0,$$

since  $N$  is normal and  $X$  is tangential. Now write any  $h^T$  on  $\partial M$  as  $h^T = h_0 + f\gamma$  where  $h_0$  is trace-free. Let  $\bar{f}$  be any smooth function and let  $\bar{h}^T = h_0 + \bar{f}\gamma$ , so that  $\bar{h}^T - h^T$  is pure-trace. Then

by the remarks following (5.2),

$$(5.4) \quad \int_{\partial M} (Ric(N, X))'_h = \int_{\partial M} (Ric(N, X))'_h.$$

Suppose first the boundary value problem in (5.1) has trivial kernel,  $K = 0$ . It follows that there exists an infinitesimal Einstein deformation  $h$  of  $(M, g)$  satisfying (5.3) with  $h^T = h_0 + f\gamma$ , for some  $f$  and with the class  $[h^T]_0 = h_1$  arbitrarily prescribed. For all such  $h$ , it follows that

$$(5.5) \quad \int_{\partial M} (Ric(N, X))'_h = 0.$$

Via (5.4), (5.5) then also holds for all  $h$ , and so by (5.2), one obtains

$$\mathcal{L}_X \tau = 0.$$

Since  $tr(\mathcal{L}_X \tau) = 0$ , this gives  $\mathcal{L}_X A = 0$  and Theorem 1.1 then follows from Proposition 2.5.

Next, suppose  $K \neq 0$ . Let  $S_0^{m,\alpha}(M)$  be the Banach space of symmetric forms on  $M$  with 0 boundary values in (5.1). Let  $K^\perp$  be the  $L^2$  orthogonal complement of  $K$  within  $S_0^{m,\alpha}(M)$ . This is a closed subspace of  $S_0^{m,\alpha}(M)$ , of finite codimension with complement  $K$ , so that  $S_0^{m,\alpha}(M) = K^\perp \oplus K$ . The operator  $\hat{L}|_{K^\perp}$  is an isomorphism onto its image  $Im(\hat{L})$ , and since  $\hat{L}$  is self-adjoint,  $Im(\hat{L}) = K^\perp$ . The kernel  $K$  is the orthogonal slice to the image, and

$$(5.6) \quad \delta(k) = 0,$$

for all  $k \in K$ .

We now construct a different linear slice  $\tilde{Q}$  to  $Im(\hat{L})$  with certain specific properties at  $\partial M$ , which are not known to hold, apriori, for  $K$ . First for each non-zero  $k \in K$ , choose a  $C^{m,\alpha}$  symmetric form  $q$  of compact support in  $M$  such that

$$(5.7) \quad \int_M \langle q, k \rangle \neq 0.$$

This gives a linear space  $Q \simeq K$ , with  $Q$  nowhere orthogonal to  $K$ , i.e. no form  $q \in Q$  is orthogonal to  $K$ , so that  $Q$  is also a slice to  $Im(\hat{L})$ . Then consider  $q + \delta^* Y$ , where  $Y$  is a solution of the equation

$$(5.8) \quad \delta \delta^* Y = -\delta(q),$$

so that  $\delta(q + \delta^* Y) = 0$ . Now one can solve the equation (5.8) with either Dirichlet, Neumann or mixed (Robin) boundary conditions at  $\partial M$ . The two boundary conditions we impose are:

$$(5.9) \quad \int_{\partial M} \langle k(N), Y \rangle = 0,$$

$$(5.10) \quad \int_{\partial M} (\delta^* Y)(N, X) = \int_{\partial M} \langle \nabla_N Y - A(Y), X \rangle = 0.$$

Here  $X$  is the given Killing field on  $\partial M$ . The first equality in (5.10) follows directly from the definition of  $\delta^* Y$  (using the fact that  $\int_{\partial M} X(f) = \int_{\partial M} f \delta(X) = 0$ , since  $X$  is Killing on  $\partial M$ ) so only the second equality is a condition.

These are mixed Dirichlet-Neumann conditions, which are straightforward to solve. In detail, consider first the homogeneous equation

$$(5.11) \quad \delta \delta^* Y = 0.$$

Let  $R_0$  be the Dirichlet-to-Robin type map sending Dirichlet data  $Y \in \chi^{m+1,\alpha}(\partial M)$  (the space of vector fields at  $\partial M$ ) to  $\nabla_N Y - A(Y)$  on  $\partial M$ , where  $Y$  solves (5.11) on  $M$ ;

$$R_0(Y) = \nabla_N Y - A(Y).$$

The map  $R_0$  is Fredholm, of Fredholm index 0, with kernel  $\mathcal{K}$  equal to the space of vector fields  $Y$  at  $\partial M$  which extend to Killing fields on  $(M, g)$ . This follows by pairing (5.11) with  $Y$  and integrating by parts. Orthogonal to the kernel,  $R_0$  is an isomorphism onto its image  $\mathcal{V}_0 \subset \chi^{m,\alpha}(\partial M)$  and  $\mathcal{V}_0 \oplus \mathcal{K} = \chi^{m,\alpha}(\partial M)$ . The Dirichlet-to-Robin map  $R_q$  for (5.8) is then an affine map onto the image

$$\mathcal{V}_q = \mathcal{V}_0 + z_q,$$

where  $z_q = \nabla_N Y$  and  $Y$  solves (5.8) with zero Dirichlet boundary data.

On the other hand, the condition (5.9) defines a codimension 1 hypersurface  $\mathcal{S} \subset \chi^{m+1,\alpha}(\partial M)$  (with normal vector  $k(N)$ ) which maps under  $R_q$  to a codimension 1 hypersurface  $R_q(\mathcal{S})$  of  $\mathcal{V}_q$ .

Suppose first  $z_q \in \mathcal{V}_0$  (e.g.  $(M, g)$  has no Killing fields) so  $\mathcal{V}_q = \mathcal{V}_0$ . We have then two codimension 1 hypersurfaces of  $\mathcal{V}_q$ , namely  $R_q(\mathcal{S})$  and the hypersurface  $\mathcal{T}_X$  defined by (5.10) with normal vector  $X$ . Any vector field  $Y$  such that  $R_q(Y)$  lies in the intersection of these two hypersurfaces satisfies (5.9)-(5.10). Since this intersection is of codimension 2 in  $\mathcal{V}_0$ , it is clear there is a large space of solutions.

If however  $z_q \notin \mathcal{V}_0$ , then  $\mathcal{V}_q$  is an affine subspace, of finite codimension in  $\chi^{m,\alpha}(\partial M)$  with  $R_q(\mathcal{S})$  of codimension 1 in  $\mathcal{V}_q$ . Let  $z'_q$  be the vector normal to  $\mathcal{V}_0$  such that  $\mathcal{V}_q = \mathcal{V}_0 + z'_q$ . Then (5.9)-(5.10) has no solutions, i.e.  $R_q(\mathcal{S}) \cap \mathcal{T}_X = \emptyset$ , if and only if the normal vector  $z'_q$  is a constant multiple of the normal vector  $X$ , so that the functional  $\int \langle \cdot, X \rangle$  is constant on  $R_q(\mathcal{S})$ . However, if  $X \in \mathcal{K}$ , then Theorem 1.1 is proved, and so, without loss of generality, one may assume  $X \perp \mathcal{K}$ , so that  $X \in \mathcal{V}_0$ . The functional  $\int \langle \cdot, X \rangle$  is then non-trivial (i.e. non-constant) and assumes the value 0 again on a codimension 2 subspace of  $\mathcal{V}_q$ . In this way, we see that (5.9) and (5.10) always have a large space of solutions.

We pick such a solution  $Y$ , for each  $q$  in a basis of  $Q$  and, extending linearly, let  $\tilde{Q} = \{\tilde{q} = q + \delta^* Y\}$ , so that  $\tilde{Q} \simeq K$ . Observe that (5.7) holds for  $\tilde{Q}$ , i.e. for any  $\tilde{q}$  there exists  $k \in K$  such that

$$(5.12) \quad \int_M \langle \tilde{q}, k \rangle \neq 0.$$

To prove this, by (5.7) it suffices to show that  $\int_M \langle \delta^* Y, k \rangle = 0$ . This follows from a standard integration-by-parts:

$$\int_M \langle \delta^* Y, k \rangle = \int_M \langle Y, \delta(k) \rangle + \int_{\partial M} \langle k(N), Y \rangle = 0,$$

where we have used (5.6) and (5.9).

Thus,  $\tilde{Q}$  is also not orthogonal to  $K$ , so gives a slice to  $Im(\hat{L})$ , and by (5.8),

$$(5.13) \quad \delta(\tilde{q}) = 0,$$

on  $M$ , for each  $\tilde{q} \in \tilde{Q}$ .

Next form the operator

$$(5.14) \quad \tilde{L}(h) = \hat{L}(h) + \pi_{\tilde{Q}}(h),$$

where  $\pi_{\tilde{Q}}$  is the orthogonal projection onto  $\tilde{Q}$ . Since by construction  $\tilde{Q}$  is linearly independent from  $Im(\hat{L})$ , it follows that  $\tilde{L}$  is an isomorphism

$$(5.15) \quad \tilde{L} : S_0^{m,\alpha}(M) \rightarrow S^{m-2,\alpha}(M).$$

Consider now the boundary value problem

$$(5.16) \quad \tilde{L}(h) = 0, \quad \delta(h) = 0, \quad [h^T]_0 = h_1, \quad H'_h = h_2.$$

Since  $\tilde{L}$  in (5.15) is a bijection, it follows from a standard substraction procedure that (5.16) has a unique solution, for arbitrary  $h_1$  and  $h_2$ . Namely, take any symmetric form  $v$  satisfying the boundary conditions in (5.16) and extend  $v$  to a smooth form on  $M$  (arbitrarily but smoothly) so

that  $\tilde{L}(v) = w$ , for some  $w$ . Let  $h_0$  be the unique solution of  $\tilde{L}(h_0) = w$  with 0-boundary values, as in (5.15). Then  $h = v - h_0$  solves (5.16). Of course solutions of (5.16) are not infinitesimal Einstein deformations in general.

Next, we claim that for any  $h_1$  and  $h_2$ , the solution  $h$  of (5.16) satisfies

$$(5.17) \quad \delta(h) = 0,$$

on  $M$ . To prove this, one has  $\hat{L}(h) = \tilde{L}(h) - \pi_{\tilde{Q}}(h) = -\pi_{\tilde{Q}}(h)$ . By (4.28), one has  $\delta\hat{L}(h) = 2\delta\delta^*(\delta(h))$  (since  $\delta\hat{L} = 0$  by the Bianchi identity) which gives

$$2\delta\delta^*(\delta(h)) = -\delta(\pi_{\tilde{Q}}(h)).$$

But  $\pi_{\tilde{Q}}(h) = \tilde{q}$  for some  $\tilde{q}$  and  $\delta(\tilde{q}) = 0$ , by (5.13). So

$$(5.18) \quad \delta\delta^*(\delta(h)) = 0,$$

on  $M$ . By assumption (in (5.16))  $\delta(h) = 0$  on  $\partial M$  and so by Lemma 2.3 (i.e. the analog of this result as mentioned in Remark 4.6) (5.17) follows.

The proof of Theorem 1.1 is now easily completed as follows. Namely for  $h$  as in (5.16),  $\delta(h) = 0$  on  $M$  and  $\hat{L}(h) = \tilde{L}(h) + \pi_{\tilde{Q}}(h) = \pi_{\tilde{Q}}(h) = \tilde{q}$ , for some  $\tilde{q}$ . We have  $\hat{L}(h) = \hat{E}'(h) + 2\delta^*\delta(h) = \hat{E}'(h)$ . Thus

$$(5.19) \quad \hat{E}'_h + \tilde{q} = 0.$$

Also  $\hat{E}'_h = Ric'_h - \frac{s'}{2}g - \frac{s}{2}h + \Lambda h$ , so that  $\hat{E}'_h(N, X) = Ric'_h(N, X) - \frac{s}{2}h(N, X) + \Lambda h(N, X)$  (since  $X$  is tangent to  $\partial M$ ). But  $Ric'_h(N, X) = (Ric(N, X))'_h - Ric(N', X) = (Ric(N, X))'_h - (\frac{s}{2} - \Lambda)\langle N', X \rangle$ . Since  $\langle N', X \rangle = -h(N, X)$ , it follows that  $\hat{E}'_h(N, X) = (Ric(N, X))'_h$  on  $\partial M$ . Recall that  $\tilde{q} = q + \delta^*Y$  where  $q$  has compact support (so vanishes near  $\partial M$ ) while  $\delta^*Y$  satisfies (5.10), i.e.

$$(5.20) \quad \int_{\partial M} (\delta^*Y)(X, N) = \int_{\partial M} \langle \nabla_N Y - A(Y), X \rangle = 0.$$

This gives

$$(5.21) \quad \int_{\partial M} (Ric(N, X))'_h = - \int_{\partial M} \tilde{q}(X, N) = 0.$$

This holds for  $[h^T]_0$  arbitrary, and so as in (5.4), (5.21) holds for all variations  $h$  on  $\partial M$ . As above via (5.2) and Proposition 2.5, this completes the proof. ■

**Remark 5.1.** The proof of Theorem 1.1 above shows that, when for instance  $H = \text{const}$  at  $\partial M$ , one has

$$(5.22) \quad K \cap Im\delta^* = 0,$$

where  $\delta^*$  acts on vector fields  $X$  tangent to  $\partial M$ , and  $K = Ker D\Pi_D$ , as in (3.5).

However, for instance in dimension 3, all Einstein deformations are pure gauge, i.e. of the form  $\delta^*V$ , for some vector field  $V$ , not necessarily tangent to  $\partial M$ , cf. Remark 3.1. Hence, if  $\Pi_D$  is degenerate at some constant curvature metric  $(M^3, g)$ , i.e.  $K = K_g \neq 0$  and again  $H = \text{const}$  at  $\partial M$ , then

$$K \cap \delta^*V \neq 0,$$

for general  $V$  at  $\partial M$ . The condition  $H = \text{const}$  is necessary here, cf. Example 4.3.

The proof of Theorem 1.1 above generalizes to the case of isometric rigidity, where the vector field  $X$  is not assumed tangent to  $\partial M$ , but is a general vector field at  $\partial M$ , preserving the mean curvature.

**Proposition 5.2.** *Let  $X$  be a vector field at  $\partial M$  generating an infinitesimal isometry at  $\partial M$  and suppose  $\mathcal{L}_X(H) = X(H) = 0$ . If  $\pi_1(M, \partial M) = 0$ , then  $X$  extends to a Killing field on  $(M, g)$ .*

**Proof:** The proof is a simple extension of the proof of Theorem 1.1. First, it is easy to see that (5.19) holds in general, without the restriction that  $X$  is tangent to  $\partial M$ , with a slight redefinition of  $\tilde{Q}$ . Namely, for general  $X$ , the left side of (5.10) remains the same, but the middle expression has new terms coming from the normal part of  $X$ ; these are of the same form as before, and their presence does not affect the validity of the proof that (5.9)-(5.10) are easily solvable for general  $X$ . Next, using (4.19)-(4.20) together with the fact that  $(\delta^* X)^T = 0$ , one obtains, for any deformation  $h$  of  $g$ ,

$$(5.23) \quad \int_{\partial M} \langle \tau'_{\delta^* X}, h \rangle = \int_{\partial M} \langle \tau'_h, (\delta^* X)^T \rangle + \int_M \langle \hat{E}'_h, \delta^* X \rangle = \int_M \langle \hat{E}'_h, \delta^* X \rangle.$$

Integrating the right-hand side by parts, and using the fact that  $\delta \hat{E}' = 0$ , it follows that

$$\int_{\partial M} \langle \tau'_{\delta^* X}, h \rangle = \int_{\partial M} (\hat{E}'_h)(X, N).$$

Now as in (5.19),  $(\hat{E}'_h)(X, N) = -\tilde{q}$ , and the analog of (5.20) holds as before. Hence,

$$(5.24) \quad \int_{\partial M} \langle \tau'_{\delta^* X}, h \rangle = 0.$$

Since the assumptions  $\mathcal{L}_X H = 0$  and  $(\delta^* X)^T = 0$  imply that  $tr \tau'_{\delta^* X} = 0$ , and since  $h$  is arbitrary at  $\partial M$  modulo pure-trace terms, it follows from (5.24) as before in the proof of Theorem 1.1 that

$$(\delta^* X)^T = 0, \quad \text{and} \quad A'_{\delta^* X} = 0.$$

at  $\partial M$ . The result then follows again from Proposition 2.5. ■

### Proof of Corollary 1.2.

Theorem 1.1 implies that the isometry group  $SO(n+1)$  of  $S^n$  extends to a group of isometries of the Einstein manifold  $(M^{n+1}, g)$ . This reduces the Einstein equations to a simple system of ODE's (the metric  $g$  is of cohomogeneity 1) and it is standard that the only smooth solutions are given by constant curvature metrics, cf. [5] for example. ■

The same proof shows that if  $(\partial M, \gamma)$  is homogeneous, then any Einstein filling metric  $(M, g)$  is of cohomogeneity 1. Such metrics have been completely classified in many situations, cf. [5] for further information.

We complete this section with a brief discussion of exterior and global boundary value problems. Thus, suppose  $M^{n+1}$  is an open manifold with compact “inner” boundary  $\partial M$  and with a finite number of ends, each (locally) asymptotically flat. Topologically, each end is of the form  $(\mathbb{R}^k \setminus B) \times T^{n+1-k}$ , or a quotient of this space by a finite group of isometries. Here  $T^{n+1-k}$  is the  $(n+1-k)$ -torus, and we assume  $3 \leq k \leq n+1$ . Assume also, as usual, that  $\pi_1(M, \partial M) = 0$ . An Einstein metric is asymptotically locally flat (ALF) if it decays to a flat metric on each end at a rate  $r^{-(k-2)}$  (the decay rate of the Green’s function for the Laplacian) where  $r$  is the distance from a fixed point.

It is proved in [3] that the analog of Theorem 2.1 holds, namely the space of asymptotically locally flat Einstein metrics on an exterior domain  $M$  is a smooth Banach manifold, for which the Dirichlet boundary map is  $C^\infty$  smooth. Lemma 2.3 also holds in this context. All of the remaining results in §2 - §5 above concern issues at or near  $\partial M$ , and it is straightforward to verify that their proofs carry over to this exterior context without change. In particular the analog of Theorem 1.1 holds:

**Proposition 5.3.** *Let  $g$  be a  $C^{m,\alpha}$  Ricci-flat metric on an exterior domain  $M$ ,  $m \geq 5$ , with a finite number of locally asymptotically flat ends. Suppose also (1.2) holds. Then any Killing field  $X$  on  $(\partial M, \gamma)$  for which  $X(H) = 0$ , extends uniquely to a Killing field on  $(M, g)$ .*

■

Next we point out that an analog of Theorem 1.1 holds for complete conformally compact Einstein metrics, where the boundary is at infinity (conformal infinity). The proof below corrects an error in the proof of this result in [2].

**Theorem 5.4.** *Let  $(M, g)$  be a conformally compact Einstein metric, with smooth conformal infinity  $(\partial M, [\gamma])$  and suppose  $\pi_1(M, \partial M) = 0$ . Then any (conformal) Killing field of  $\partial M$  extends to a Killing field of  $(M, g)$ .*

**Proof:** The proof is a simple adaptation of the proof of Theorem 1.1, using information provided in [2], to which we refer for some details. Let  $t$  be a geodesic compactification of  $(M, g)$  and let  $S(t)$  and  $B(t)$  be the level and super-level sets of  $t$ , so that  $\partial M = S(0)$ ,  $M = B(0)$ . The Killing vector field  $X$  on  $(\partial M, \gamma)$  is extended into  $M$  to be in Bianchi gauge, so that  $\delta^* X$  is transverse-traceless. One then has  $\langle X, N \rangle = O(t^{n+1})$ , where  $N = -t\partial_t$  is the unit outward normal at  $S(t)$  (cf. [2]). As in (5.23)-(5.24) (setting  $M = B(t)$ ) one then has

$$(5.25) \quad \frac{1}{2} \int_{S(t)} \langle \mathcal{L}_X \tau, h \rangle = \int_{S(t)} \langle \tau'_h, (\delta^* X)^T \rangle + \int_{B(t)} \langle \hat{E}'_h, \delta^* X \rangle = \int_{S(t)} \langle \tau'_h, (\delta^* X)^T \rangle.$$

Here we note that  $\delta^* X = O(t^n)$  while  $X(H) = O(t^{n+1})$  on  $S(t)$ . The equation (5.25) holds for any  $h$  such that  $\tilde{L}(h) = 0$  satisfying the boundary conditions  $\delta(h) = 0$ ,  $[h^T]_0 = h_1$ ,  $H'_h = h_2$  with  $h_1, h_2$  arbitrary on  $S(t)$ . This follows from the discussion above concerning (5.16), which holds without the assumption that  $X$  is Killing on the boundary  $S(t)$ . As in [2], let  $\tilde{\kappa} = t^{-n} \delta^* X$ , so that  $\tilde{\kappa}$  is uniformly bounded as  $t \rightarrow 0$  and  $\tilde{\kappa}(N, \cdot) = O(t)$ ,  $tr \tilde{\kappa} = 0$ .

We then choose  $h$  such that  $[h^T]_0 = [\tilde{\kappa}^T]_0$  on  $S(t)$ . Substituting this in (5.25) gives

$$(5.26) \quad \int_{S(t)} \langle \mathcal{L}_X \tau, \tilde{\kappa} \rangle = 2 \int_{S(t)} \langle \tau'_{\tilde{\kappa}}, (\delta^* X)^T \rangle.$$

However, a straightforward computation, given in [2] shows that

$$\int_{S(t)} \langle \mathcal{L}_X \tau, \tilde{\kappa} \rangle = -\frac{1}{4}(3n-2) \int_{\partial M} |\mathcal{L}_X g_{(n)}|^2 dV_\gamma + o(1),$$

where  $g_{(n)}$  is the  $n^{\text{th}}$  term in the Fefferman-Graham expansion of  $(M, g)$ , while

$$2 \int_{S(t)} \langle \tau'_{\tilde{\kappa}}, (\delta^* X)^T \rangle = -\frac{1}{4}(2n-2) \int_{\partial M} |\mathcal{L}_X g_{(n)}|^2 dV_\gamma + o(1).$$

It follows that on  $\partial M$ ,

$$\mathcal{L}_X g_{(n)} = 0,$$

so that the flow of  $X$  preserves both the boundary metric  $\gamma$  and the  $g_{(n)}$  term. The result then follows from the unique continuation result, [2, Corollary 4.4], analogous to Theorem 2.2.

■

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